Semiconductor Physics, Quantum Electronics & Optoelectronics, 2016. V. 19, N 1. P. 28-33.
doi: 10.15407/spqeo19.01.028

PACS 42.65.Wi, 77.22.Ch

Search of mode wavelengths in planar waveguides by using Fourier transform of wave equation

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Abstract. The article describes a numerical method based on Fourier transform for studying propagating optical waves in dielectric planar waveguides. The inverse problem to the known direct one in waveguide investigation is proposed, namely a search of light wavelengths according to taken values of propagation constants. For each constant a set of wavelengths is obtained, among which an input value of wavelength from direct problem exists necessarily. A high accuracy of the method proposed is confirmed by exact values obtained by solution of transcendental dispersion equation. This method is tested on many examples, in particular, for waveguides of different permittivity profiles or for TE and TM modes propagate there.

Keywords: Fourier transform, eigenvalues, permittivity, propagation constant, spatial frequency, wave equation.

Manuscript received 14.10.15; revised version received 12.01.16; accepted for publication 16.03.16; published online 08.04.16.

1. Introduction

Dielectric planar waveguides are the structures used to limit and control light in waveguide devices and integrated optics circuits [1]. One of the most common applications of such waveguides today is semiconductor distributed feedback micro-lasers [2-4] using them as an active layer. These lasers provide high speed data transfer, and single-mode generation is achieved by feedback filter in a form of corrugated layer. The filter is formed inside waveguide semiconductor structure parallel to the active layer.

For designing the devices based on planar waveguides, it is necessary to know propagation constants of waveguide modes that correspond to a taken wavelength. A number of approximate methods are used to determine propagation constants of localized modes in gradient planar waveguides [5, 6], which for the first time have been developed for analyzing the problems of quantum mechanics. As the structure of wave equation for modes of TE polarization is similar to that of the stationary Schrödinger equation in quantum mechanics, it is possible to use its analytical methods, particularly WKB (Wentzel–Kramers–Brillouin) approximation [6], for studying the planar waveguides.

A typical permittivity distribution of symmetric gradient waveguide is shown in Fig. 1, where ε_0 is the substrate permittivity, ε_1 is the maximum value of permittivity in the active layer. For some profiles of waveguide permittivity $\varepsilon(x)$, the accurate analytical solutions are found [7-9]. As in a waveguide $\varepsilon_1 > \varepsilon_0$, propagation of a localized guided mode with the propagation constant β is possible, and the electric field distribution is described by the following function: $E(x,z) = E(x)\exp(-i\beta z)$, where x and z are the transverse and longitudinal coordinates, respectively, $E(x)$ is the electric field amplitude, i – imaginary unit. But, even in the simplest case (a step profile of permittivity as particular case of the gradient one), a search of propagation constants leads to solution of transcendental algebraic equation [10]. Problem becomes more difficult, if the permittivity varies according to a complex function along the axis x. Well-known methods to find propagation constants and waveguide mode fields are mostly analytical, too cumbersome, and their accuracy is rather low. For example, WKB approximation allows to calculate with a high accuracy these propagation constants that correspond to field distributions with a large number of nodes (points where electromagnetic field is zero). Mostly, in planar waveguides situation is somewhat different: it is necessary to find the values of propagation constants for the lowest modes with a small number of nodes (1st, 2nd order modes) or without them (basic mode) in appropriate electromagnetic field.

The current state of computer hardware and software sophistication allows using the numerical methods effectively to search propagation constants and fields of gradient planar waveguides. A numerical method of wave equation solution for planar waveguides in a coordinate domain is known. In this method, the second derivative by coordinate is replaced by the difference operator, and ultimately the problem is reduced to a solution of the eigenvalue/eigenvector problem [11, 12]. But this method is characterized by low accuracy, because electromagnetic field may occupy a large spatial range, especially for waveguide modes with large indices (higher order modes).

In recent years, the numerical method for finding the propagation constants based on the wave equation Fourier transform was developed [13], and it is characterized by high accuracy of analysis. By this method, it is possible to find all the propagation constants of localized modes and appropriate discrete Fourier transforms of field distribution in a waveguide in one calculation cycle [14]. The method was tested on many gradient waveguides. For example, let waveguide permittivity is described by a function $\varepsilon(x) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)/\cosh^2(2x/d)$ (see the figure), where d is the thickness of active layer. Then for this waveguide, an exact analytical solution exists, and exact values of propagation constants are found [1, 7], which are listed in the left column of Table 1 for a waveguide with the following parameters: $\varepsilon_0 = 2.25$, $\varepsilon_1 = 2.89$, $d = 5$ μm. Electro-magnetic wave of the length $\lambda = 1$ μm propagates in the structure. At this wavelength, the waveguide has 13 guided modes. Values shown in the right column of Table 1 are the propagation constants calculated using the numerical method described in [14].

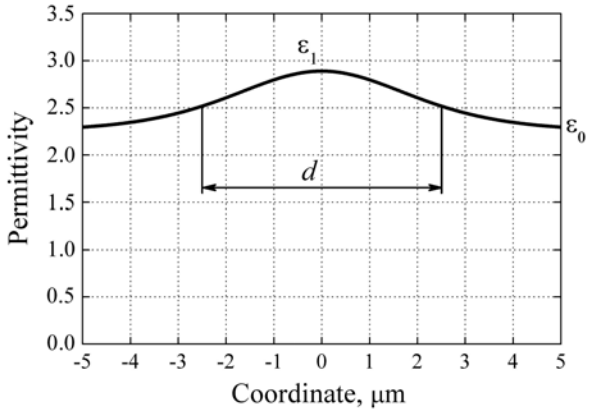

Image of permittivity distribution for a symmetric gradient waveguide.

The propagation constants of waveguide modes with indices from 0 to 11 calculated by both methods are the same, except the last one, appropriate fields of which have a maximum length in the coordinate space; so for them a small error is available. This numerical method provides high calculation accuracy and, as the research shows, it is characterized by high numerical stability. A search of propagation constants by using the numerical method [13, 14] is reduced to the problem on eigenvalues (square of the propagation constants) and eigenvectors (field discrete Fourier transforms in a waveguide) that look like $\mathbf{MV} = \beta^2\mathbf{V}$, where $\mathbf{M}$ is some square matrix depending on the parameters mentioned above, $\mathbf{V}$ is the vector-column, the elements of which are actually eigenvectors.

It often happens that one needs to solve the inverse problem, *i.e.*, for a planar waveguide with certain parameters the propagation constant of some guided mode is known, and it is necessary to find the wavelength that corresponds to the taken propagation constant. This problem arises in the analysis of

Table 1. Values of propagation constants (μm^{-1}) in a gradient waveguide obtained by two methods.

Index of propagation constant	Exact method [1]	Numerical method [14]
0	10.59058151	10.59058151
1	10.41422086	10.41422086
2	10.25044271	10.25044271
3	10.09985917	10.09985917
4	9.96306855	9.96306855
5	9.84064604	9.84064604
6	9.73313382	9.73313382
7	9.64103073	9.64103073
8	9.56478192	9.56478192
9	9.50476895	9.50476895
10	9.46130077	9.46130077
11	9.43460608	9.43460608
12	9.42482808	9.42482739

waveguide distributed feedback lasers by using the coupled wave method [2]. In the course of this analysis, all localized modes at which generation is possible are determined [15]. So, the aim of the work is to show how this problem can be solved successfully by using the numerical method based on Fourier transform of wave equation. The task is again reduced to another eigenvalue/eigenvector problem where square of wavelengths are eigenvalues: $\mathbf{M_1V} = \lambda^2\mathbf{M_2V}$, where $\mathbf{M_1}$ and $\mathbf{M_2}$ are the square matrix with the dimension defined by necessary accuracy of calculations; λ is the sought wavelength, **V** – vector-column corresponding to the Fourier transform of field distribution.

2. One-dimensional wave equations and their Fourier transforms

If electric field is perpendicular to the plane of incident wave on the interface of waveguide film and substrate, i.e., $\mathbf{E} = \{0, E_y, 0\}$, in this structure the transverse electric modes (TE modes) are formed, for which the wave equation is written as [15]

$$\frac{d^2E(x)}{dx^2} + \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon(x)E(x) = \beta^2 E(x). \quad (1)$$

If electric field is parallel to the plane of incident wave, i.e., $\mathbf{H} = \{0, H_y, 0\}$, this case corresponds to the transverse magnetic modes (TM modes), and wave equation will look like:

$$\frac{d^2H(x)}{dx^2} - \frac{d\ln\varepsilon(x)}{dx}\frac{dH}{dx} + \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon(x)H(x) = \beta^2 H(x). \quad (2)$$

Functions $E(x)$, $H(x)$ describing fields in waveguide localized modes and their first derivatives tend towards zero at $x \to \pm\infty$. That is why, for these functions, their first and second derivatives the Fourier transform exists. Function integrity in (1) and (2) is an important aspect of this approach. As the expressions for field components of appropriate modes are identical, it is sufficient to introduce the Fourier transforms only for one of them, *e.g.*, for $E(x)$ [16]:

$$\int_{-\infty}^{\infty} E(x)\exp(-i2\pi ux)dx = E(u), \quad (3)$$

$$\int_{-\infty}^{\infty} \frac{dE(x)}{dx}\exp(-i2\pi ux)dx = i2\pi uE(u), \quad (4)$$

$$\int_{-\infty}^{\infty} \frac{d^2E(x)}{d^2x}\exp(-i2\pi ux)dx = -(2\pi u)^2 E(u), \quad (5)$$

where u is the spatial frequency, $E(u)$ – Fourier transform of electric field.

Besides, for functions for which the Fourier transforms exist, *i.e.*, $F\{g(x)\} = G(u)$, $F\{h(x)\} = H(u)$, the following equation is yet right [16]:

$$F\{g(x)h(x)\} = \int_{-\infty}^{\infty} G(u-v)H(v)dv, \quad (6)$$

where $F\{\ldots\}$ is the Fourier transform. Equation (6) describes the convolution theorem.

One takes the Fourier transforms of left and right parts of (1) and (2) taking into account the expressions (3) to (6). As a result, the transition from differential equations to integral ones occurs, and we obtain next wave equations in a frequency domain:

$$-4\pi^2u^2E(u) + \left(\frac{2\pi}{\lambda}\right)^2 \int_{-\infty}^{\infty} \varepsilon(u-v)E(v)dv = \beta^2 E(u), \quad (7)$$

$$\begin{aligned} &-4\pi^2u^2H(u) - 2i\pi u \int_{-\infty}^{\infty} F\left\{\frac{d\ln\varepsilon(x)}{dx}\right\}(u-v)vH(v)dv + \\ &+\left(\frac{2\pi}{\lambda}\right)^2 \int_{-\infty}^{\infty} \varepsilon(u-v)H(v)dv = \beta^2 H(u). \end{aligned} \quad (8)$$

The Fourier transform of permittivity is

$$\begin{aligned} \varepsilon(u) &= F\{\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)f(x)\} = \\ &= \varepsilon_0\,\delta(u) + (\varepsilon_1 - \varepsilon_0)F\{f(x)\}, \end{aligned}$$

where $\delta(u)$ is the Dirac delta function in a spatial frequency domain, $f(x)$ – function describing the permittivity distribution. Besides, the following condition is imposed on the function $f(x)$:

$$\int_{-\infty}^{\infty} |f(x)|dx < M, \quad (9)$$

where M is a finite number.

3. A search of wavelengths according to the taken propagation constant

To demonstrate a way of finding the wavelengths corresponding to the taken propagation constant β, let's consider the equation (7) in another form:

$$4\pi^2 \int_{-\infty}^{\infty} \varepsilon(u-v)E(v)dv = \lambda^2\left[\beta^2 + 4\pi^2u^2\right]E(u). \quad (10)$$

In (10), one can replace the integral by sum and go to the equation in a discrete form. By changing continuous values of u and v by the discrete ones, we obtain:

$$\begin{aligned} &4\pi^2 \sum_{k=-(N-1)/2}^{(N-1)/2} \varepsilon(s\Delta - k\Delta)E(k\Delta)\Delta = \\ &= \lambda^2\left[\beta^2 + 4\pi^2(s\Delta)^2\right]E(s\Delta), \end{aligned} \quad (11)$$

where N is the number of points in which electric field is sought; s and k are the indices on which summation is

Semiconductor Physics, Quantum Electronics & Optoelectronics, 2016. V. 19, N 1. P. 28-33.
doi: 10.15407/spqeo19.01.028

done: $s, k \le |(N-1)/2|$; Δ is the partitioning step of maximum spatial frequency $u_{\max}$: $\Delta = u_{\max}/N$. Value of N should be taken large enough and unpaired, while its ratio with the frequency $u_{\max}$ should give an optimum value of partitioning step Δ. It is also assumed that the function $E(s\Delta)$ is sought in the interval from $-u_{\max}/2$ to $u_{\max}/2$, and beyond the field attenuates very quickly, *i.e.*, $E(u > |u_{\max}/2|) \to 0$.

One can write the last equation for all discrete spatial frequencies $u_s = s\Delta$. Then, a set of these equations will be written in a matrix form where the value of square wavelength λ^2 is common to all values of index s:

$$\mathbf{M_1 V} = \lambda^2 \mathbf{M_2 V}, \quad (12)$$

where $\mathbf{M_1}$ is the square symmetric matrix of elements $4\pi^2\varepsilon(s\Delta - k\Delta)$, $\mathbf{M_2}$ – diagonal matrix of elements $\beta^2 + 4\pi^2(s\Delta)^2$ in the main diagonal, $\mathbf{V}$ – vector-column of elements $E(s\Delta)$.

So, the problem is reduced to the problem on eigenvalues (square wavelength) and eigenvectors (discrete Fourier transform of field $E(x)$) which correspond to the found value of λ^2. By carrying out the inverse discrete Fourier transform of eigenvector, we obtain field distribution along coordinate x for every value of wavelength corresponding to the appropriate localized mode.

Propagation constants β_v of symmetric planar waveguide for a taken wavelength λ satisfy the following inequality: $\frac{2\pi n_0}{\lambda} < \beta_v < \frac{2\pi n_1}{\lambda}$, where $n_0 = \sqrt{\varepsilon_0}$ and $n_1 = \sqrt{\varepsilon_1}$ are the refractive indices of substrate and active layer, respectively. For the inverse problem, the wavelengths λ_v must satisfy the following inequality accordingly to the known propagation constant β:

$$\frac{2\pi n_0}{\beta} < \lambda_v < \frac{2\pi n_1}{\beta}. \quad (13)$$

For all the propagation constants from Table 1, matching sets of wavelengths are found using the matrix equation (12) and inequality (13). In every set, the wavelength $\lambda = 1$ µm is available, which confirms correctness of calculations. If we take an arbitrary wavelength from all sets and use the equation $\mathbf{MV} = \beta^2\mathbf{V}$, we find appropriate propagation constant among the set obtained again. A solution of both direct and inverse problems was carried out using the following numerical process parameters: number of points $N = 2001$, maximum spatial frequency $u_{\max} = 10$ µm^{-1}. They are selected from the subject to the Whittaker-Shannon sampling theorem [16].

The permittivity of investigated waveguides is described by functions of the following type: $\varepsilon(x) = 2.25 + 0.64\,f(2x/d)$, where $d = 5$ µm. One can check easily that all the functions $f(2x/d)$ presented in Tables 2 and 3 satisfy the condition (9). In these tables, several examples for the waveguide permittivity profiles are shown, so there is a possibility to check a direct problem by using the inverse one. For the direct problem, calculations are carried out at the light wavelength $\lambda = 1$ µm, for the inverse one they are done using the propagation constants with the indices 0, 1, 2, $m-2$, $m-1$ and m from Table 2, where m is the last number of β (it corresponds to the smallest propagation constant according to its value).

Table 2. Propagation constants (µm^{-1}) for several waveguides calculated for the wavelength 1 µm.

Index of propagation constant	Parabolic profile	Gauss profile	Exponential profile
0	10.58687909	10.52227751	10.47134576
1	10.39522986	10.21638483	10.21797941
2	10.20000124	9.94381586	10.06583391
3	10.00114042	9.71247251	9.93737112
4	9.79935523	9.53409763	9.83825131
5	9.59914463	9.43166891	9.75205958
6	9.43189030	For this waveguide, 6 guided modes exist	9.68270923
…	For this waveguide, 7 guided modes exist		…
13			9.43890608
14			9.42980648
15			9.42530846
			For this waveguide, 16 guided modes exist

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Table 3. Light wavelength (μm) obtained via equations (12) and (13) for the propagation constants from Table 2.

Index of propagation constant	Parabolic profile	Gauss profile	Exponential profile
0	0.99999999946	1.00000000000	0.99999999978
1	0.99999999944	0.99999999998	1.00000000018
2	0.99999999965	0.99999999959	0.99999999999
$m-2$	0.99999999930	0.99999999948	0.99999999935
$m-1$	0.99999999909	1.00000000016	1.00000000007
m	0.99999999953	0.99999999983	1.00000000024

It is seen that for the taken parameters of waveguide and numerical process, the largest number of guided modes ($n = 16$) will be generated in the structure with an exponential profile of permittivity, and the smallest one ($n = 6$) will be in the waveguide with a Gauss distribution.

Obviously, for any propagation constant from Table 2, among the set of wavelengths found via equations (12) and (13), it should be the wavelengths very close to $\lambda = 1$ μm. The wavelength values for 6 different modes (3 lowest and 3 highest indices) are adduced in Table 3.

Our analysis of results from Table 3 shows that $|\lambda - 1| < 10^{-9}\,\mu\text{m}$, *i.e.*, the relative error of computations is less than 10^{-9}. Therefore, for this problem accuracy of calculations by using the proposed numerical method is extremely high.

4. Conclusion

The results showed that the inverse problem of a search of waveguide mode wavelengths corresponding to the taken propagation constant is solved with sufficiently high accuracy. This method is use comfortably for planar gradient waveguides and for complex ones consisting of several layers with certain thicknesses and refractive indices. Mainly, these waveguides are used in semiconductor lasers [15].

Accuracy of computations is defined by numerical process parameters N and $u_{\max}$, *i.e.*, these parameters should have such values that the sampling theorem [16] is practically performed. In this case, high accuracy of solution can be provided. One can see from Table 1 that the propagation constant of a waveguide mode with the highest index is defined with the least accuracy. For this mode, the electric field decreases slowly with increasing the coordinate x. Therefore, N value should be taken large enough, and Δ value should be done small, which can be provided by a small spatial frequency $u_{\max}$. On the other hand, $u_{\max}$ cannot be taken quite small, as in this case values of $E(\pm u_{\max}/2)$ and $\varepsilon(\pm u_{\max}/2)$ will be different from zero. To select the maximum spatial frequency, one can use the criterion proposed in [17]. According to it, some function I on $u_{\max}$ is sought using the rule:

$$I(u_{\max}) \int_{-0.5u_{\max}}^{0.5u_{\max}} |F\{f(x)\}|^2 du\,, \tag{14}$$

where $F\{f(x)\}$ is the Fourier transform of function $f(x)$.

This integral goes to saturation at increase of $u_{\max}$, so the bottom boundary of acceptable values can be determined from the condition that $I(u_{\max})$ is virtually unchanged at change of spatial frequency. If these conditions are provided, by the method proposed one can find the solutions of both direct and inverse problems that concern planar gradient waveguides with high accuracy.

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