Optics

Implicit quiescent optical solitons for perturbed Fokas–Lenells equation with nonlinear chromatic dispersion and a couple of self-phase modulation structures by Lie symmetry

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Abstract. The paper retrieves implicit quiescent optical solitons to the perturbed Fokas–Lenells equation that is considered with nonlinear chromatic dispersion and a couple of self-phase modulation structures. They are quadratic-cubic and quadratic-cubic-quartic forms along with their respective generalized counterparts. The results from linear temporal evolution as well as generalized temporal evolution formats are presented. Lie symmetry analysis is the integration tool implemented in the work.

Keywords: solitons, Lie symmetry, Appell hypergeometric function, elliptic integral, quadrature.

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1. Introduction

The study of quiescent optical solitons with nonlinear chromatic dispersion (CD) has gained momentum during the last couple of decades [1–11]. There are several models that have been addressed in this context. A few of them are the nonlinear Schrödinger's equation, Lakshmanan–Porsezian–Daniel model, complex Ginzburg–Landau equation and many others. There are several forms of self-phase modulation (SPM) structures that were incorporated in such studies. In all cases, it was observed that the soliton mobility gets stalled with the presence of the nonlinear CD. The soliton solutions that emerged from the analysis are of various forms. Some of them were explicit bright or singular solitons while others were characterized in terms of quadratures. Many forms are implicit quiescent optical solitons.

There are several forms of non-Kerr laws of nonlinear SPM structure that were taken into consideration. The current paper addresses the retrieval of implicit quiuescent optical solitons from the perturbed Fokas–Lenells equation (FLE) that is studied with two forms of SPM and their respective generalizations. They are the quadratic-cubic and quadratic-cubic-quartic nonlinear structures. The perturbation terms are of Hamiltonian type and come with arbitrary intensity. These reflect self-steepening effect, soliton self-frequency shift and the inter-modal dispersion. The Lie symmetry analysis is employed to recover the implicit quiescent optical solitons. In a couple of cases, the results are in terms of quadratures. All of them are exhibited in the rest of the paper with the introduction of the structure of the model at each stage.

2. Fokas–Lenells equation

2.1. Linear temporal evolution

The dimensionless form of the FLE in presence of perturbation terms with nonlinear CD and non-Kerr law of SPM as well as linear temporal evolution is given as

$$iq_t + a(|q|^n q)_{xx} + F(|q|^2)q + i\sigma|q|^2 q_x = i[\alpha q_x + \lambda(|q|^2 q)_x + \mu(|q|^2)_x q]. \quad (1)$$

Here, in Eq. (1), the dependent variable $q(x,t)$ is a complex-valued function and represents the amplitude of a wave that propagates through an optical fiber.

The independent variables are x and t that stand for the spatial and temporal coordinates, respectively. The first term in (1) represents the linear temporal evolution with $i = \sqrt{-1}$. The coefficient a is the nonlinear chromatic dispersion and n is its nonlinearity parameter. The parameter σ accounts for the nonlinear dispersion. In the perturbation terms on the right hand side, α represents the intermodal dispersion, λ accounts for self-steepening, and μ gives the self-frequency shift. The functional F represents non-Kerr law nonlinearity that comes from the refractive index change.

In order to analyze Eq. (1), the following substitution is selected:

$$q(x,t) = \phi(x)\mathrm{e}^{i\omega t} \tag{2}$$

where $\phi(x)$ represents the amplitude component of the quiescent soliton, and ω represents the wave number of the soliton. Substituting (2) into (1), the two components from the real and imaginary parts are given by the following ordinary differential equations (ODEs):

$$a(n+1)\phi^n(x)[n\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + F\{\phi^2(x)\}\phi(x) - \omega\phi^2(x) = 0, \tag{3}$$

and

$$\phi(x)\phi'(x)\{\alpha + \phi^2(x)(3\lambda + 2\mu - \sigma)\} = 0, \tag{4}$$

respectively. For integrability, one is compelled to choose from (4):

$$\alpha = 0, \tag{5}$$

and

$$\lambda = \frac{1}{3}(\sigma - 2\mu). \tag{6}$$

This shows that the perturbed FLE that needs to be studied in the paper cannot contain the inter-modal dispersion term for any form of SPM. Thus by virtue of (5) and (6), Eq. (1) must reduce to

$$iq_t + a(|q|^n q)_{xx} + F(|q|^2)q + i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q]. \tag{7}$$

Eq. (7) with the real part represented by (3) will be studied in the rest of this paper for the range of SPM structures.

2.2. Generalized temporal evolution

For generalized temporal evolution, the FLE takes the following form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + F(|q|^2)q^l + i\sigma|q|^2(q^l)_x = i[\alpha(q^l)_x + \lambda(|q|^2 q^l)_x + \mu(|q|^2)_x q^l]. \tag{8}$$

In (8), the parameter l accounts for the generalized temporal evolution. At $l = 1$, Eq. (8) collapses to the model with linear temporal evolution given by (1). The starting point is the hypothesis given by (2). Thus, the real part gives

$$a(l+n)\phi^n(x)[\phi(x)\phi''(x) + (l+n-1)\{\phi'(x)\}^2] + F\{\phi^2(x)\} - l\omega\phi^2(x) = 0, \tag{9}$$

and

$$\phi(x)\phi'(x)[\alpha l + \{\phi'(x)\}^2\{(l+2)\lambda - l\sigma + 2\mu\}] = 0. \tag{10}$$

The imaginary part of Eq. (10) reveals the expression (5) and a generalized version of the expression (6) as

$$\lambda = \frac{l\sigma - 2\mu}{l+2}. \tag{11}$$

Thus, by virtue of (5) and (11), the governing model (8) modifies to

$$i(q^l)_t + a(|q|^n q^l)_{xx} + F(|q|^2)q^l + i\sigma|q|^2(q^l)_x = \frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l]. \tag{12}$$

For the case with included generalized temporal evolution, the governing model must be free from intermodal dispersion for its integrability. Eq. (12), along with the reduced ODE given by (9), will be analyzed for the two forms of SPM and their generalized versions by Lie symmetry in the subsequent sections.

3. Quadratic-cubic nonlinearity

For quadratic-cubic nonlinearity, the SPM structure is

$$F(|q|^2) = b_1|q| + b_2|q|^2, \tag{13}$$

where b_1 and b_2 are real-valued constants. This quadratic-cubic form of SPM will now be applied to FLE with nonlinear CD to retrieve its quiescent optical solitons for linear and generalized temporal evolutions.

3.1. Linear temporal evolution

For the SPM structure given by (13), the governing model (1) transforms to

$$iq_t + a(|q|^n q)_{xx} + (b_1|q| + b_2|q|^2)q + i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q]. \tag{14}$$

Thus, the corresponding ODE given by (3), after substituting (2) in it takes the following form:

$$a(n+1)\phi^n(x)[n\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_1\phi^3(x) + b_2\phi^4(x) - \omega\phi^2(x) = 0. \tag{15}$$

The above equation admits a single Lie point symmetry, namely $\partial/\partial x$. This symmetry, when implemented into the integration process, leads to the implicit solution in terms of Appell hypergeometric function of two variables as follows:

$$x = \pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(n+1)(n+2)}{\omega}} \times F_1\left(\frac{n}{2};\frac{1}{2},\frac{1}{2};\frac{n+2}{2};-\frac{2(n+2)(n+3)\phi b_2}{B_1+B_2},-\frac{2(n+2)(n+3)\phi b_2}{B_1-B_2}\right), \tag{16}$$

where

$$B_1 = b_1(n+2)(n+4), \tag{17}$$

and

$$B_2 = \sqrt{(n+2)(n+4)\{4b_2(n+3)^2\omega + b_1^2(n+2)(n+4)\}}. \tag{18}$$

The solution (16) introduces the constraint on the parameters as

$$a\omega > 0. \tag{19}$$

Here, the Appell hypergeometric function of two variables is defined by the infinite series:

$$F_1(a; b_1, b_2; c; x, y) = x^m y^n \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m! n!}\right), \quad (20)$$

which is convergent inside the region

$$\max(|x|, |y|) < 1. \quad (21)$$

In this case, (16) implies

$$\max\left(\left|\frac{\phi b_2}{B_1+B_2}\right|, \left|\frac{\phi b_2}{B_1-B_2}\right|\right) < 1. \quad (22)$$

3.2. Generalized temporal evolution

For generalized temporal evolution, Eq. (12) reduces to

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q| + b_2|q|^2)q^l + i\sigma|q|^2(q^l)_x = \frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l], \quad (23)$$

and with the implemented hypothesis (2) leads to the following ODE for $\phi(x)$:

$$a(l+n)\phi^n(x)[\phi(x)\phi''(x) + (l+n-1)\{\phi'(x)\}^2] + b_1\phi^3(x) + b_2\phi^4(x) - l\omega\phi^2(x). \quad (24)$$

Eq. (24) admits a single Lie point symmetry, namely $\partial/\partial x$. This symmetry, being implemented, yields the following implicit solution in terms of Appell hypergeometric function:

$$x = \pm \frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(l+n)(2l+n)}{l\omega}} = F_1\left(\frac{n}{2}; \frac{1}{2}, \frac{1}{2}; \frac{2+n}{2}; -\frac{2(2l+n)(1+2l+n)\phi b_2}{B_1+B_2}, -\frac{2(2l+n)(1+2l+n)\phi b_2}{B_1-B_2}\right) \quad (25)$$

where

$$B_1 = b_1(2l+n)(2l+n+2), \quad (26)$$

and

$$B_2 = \sqrt{(2l+n)(2l+n+2) \times \{4b_2 l\omega(2l+n+1)^2 + b_1^2(2l+n)(2l+n+2)\}}. \quad (27)$$

In this case, Eq. (25) introduces an additional constraint given by

$$al\omega > 0. \quad (28)$$

The convergence criteria (20) leads to the same criterion as given by (22).

4. Generalized quadratic-cubic nonlinearity

In this case, the SPM structure takes the following form:

$$F(|q|^2) = b_1|q|^m + b_2|q|^{2m}, \quad (29)$$

where b_1 and b_2 are real-valued constants and m is a real number. This generalized quadratic-cubic form of SPM applied to FLE for nonlinear CD. The resulting models will be integrated to retrieve the quiescent optical solitons for linear and generalized temporal evolutions.

4.1. Linear temporal evolution

For the SPM structure given by (29), the governing model (7) takes the form:

$$iq_t + a(|q|^n q)_{xx} + (b_1|q|^m + b_2|q|^{2m})q + i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q]. \quad (30)$$

Thus, the corresponding ODE given by (4), by virtue of (2) in this case, reduces to

$$a(n+1)\phi^n(x)[n\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_1\phi^{m+2}(x) + b_2\phi^{2m+2}(x) - \omega\phi^2(x) = 0, \quad (31)$$

Eq. (31) permits the same single Lie point symmetry as before, which leads to its integral to implicit quiescent optical solitons as

$$x = \pm \frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(n+1)(n+2)}{\omega}} \times F_1\left(\begin{matrix}\frac{n}{2m}; \frac{1}{2}, \frac{1}{2}; 1+\frac{n}{2m}; \\ -\frac{2(n+2)(m+n+2)\phi^m b_2}{B_1-B_2}, -\frac{2(n+2)(m+n+2)\phi^m b_2}{B_1+B_2}\end{matrix}\right), \quad (32)$$

where

$$B_1 = b_1(n+2)(2m+n+2), \quad (33)$$

and

$$B_2 = \sqrt{(n+2)(2m+n+2) \times \{4b_2\omega(m+n+2)^2 + b_1^2(n+2)(2m+n+2)\}}. \quad (34)$$

Here, the parameter constraint (19) remains the same, while the convergence criterion (22) becomes

$$\max\left(\left|\frac{\phi^m b_2}{B_1+B_2}\right|, \left|\frac{\phi^m b_2}{B_1-B_2}\right|\right) < 1. \quad (35)$$

4.2. Generalized temporal evolution

For generalized temporal evolution, the FLE takes the form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^m + b_2|q|^{2m})q^l + i\sigma|q|^2(q^l)_x = \frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l]. \quad (36)$$

The starting point is the same hypothesis as given by (2). Thus the real part gives

$$a(l+n)\phi^n(x)[\phi(x)\phi''(x) + (l+n-1)\{\phi'(x)\}^2] + b_1\phi^{m+2}(x) + b_2\phi^{2m+2}(x) - l\omega\phi^2(x) = 0. \quad (37)$$

With the same implemented translational Lie symmetry as before, one arrives at the following implicit quiescent optical soliton:

$$x = \pm \frac{\phi^{\frac{n}{2}}}{n} \sqrt{\frac{2a(l+n)(2l+n)}{l\omega}} \times F_1\left(\begin{matrix} \frac{n}{2m}; \frac{1}{2}, \frac{1}{2}; 1 + \frac{n}{2m}; \\ -\frac{2(2l+n)(2l+m+n)\phi^m b_2}{B_1 - B_2}, -\frac{2(2l+n)(2l+m+n)\phi^m b_2}{B_1 + B_2} \end{matrix} \right), \quad (38)$$

where

$$B_1 = b_1(2l+n)\{2(l+m)+n\}, \quad (39)$$

and

$$B_2 = \sqrt{\begin{matrix} (2l+n)\{2(l+m)+n\} \times \\ [4b_2 l\omega(2l+m+n)^2 + b_1^2(2l+n)\{2(l+m)+n\}]. \end{matrix}} \quad (40)$$

The constraint conditions (28) and (35) are to hold here to ensure existence of the solutions.

5. Quadratic-cubic-quartic nonlinearity

In this case, the SPM is structured as

$$F(|q|^2) = b_1|q| + b_2|q|^2 + b_3|q|^3, \quad (41)$$

where b_j for $1 \le j \le 3$ are real-valued constants. Two subsections that follow present linear and generalized temporal evolutions.

5.1. Linear temporal evolution

The FLE with such an SPM takes the form:

$$\begin{aligned} &iq_t + a(|q|^n q)_{xx} + (b_1|q| + b_2|q|^2 + b_3|q|^3)q + \\ &i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q]. \end{aligned} \quad (42)$$

In this case, using the hypothesis (2), the real part equation gives

$$\begin{aligned} &a(n+1)\phi^n(x)[n\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + \\ &b_1\phi^2(x) + b_2\phi^3(x) + b_3\phi^4(x) - \omega\phi^2(x) = 0. \end{aligned} \quad (43)$$

For integrability of Eq. (43), one needs to choose

$$n = 1. \quad (44)$$

Therefore, Eqs. (42) and (43) respectively transform to

$$\begin{aligned} &iq_t + a(|q|q)_{xx} + (b_1|q| + b_2|q|^2 + b_3|q|^3)q + \\ &i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q], \end{aligned} \quad (45)$$

and

$$\begin{aligned} &2a\phi^n(x)[\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_1\phi^2(x) + \\ &b_2\phi^3(x) + b_3\phi^4(x) - \omega\phi^2(x) = 0. \end{aligned} \quad (46)$$

With the translational Lie symmetry applied to (46), one recovers the implicit solution in terms of elliptic integral of the first kind as

$$x = \pm\sqrt{-\frac{240a(\phi\kappa_1 - 1)(\phi\kappa_2 - 1)(\phi\kappa_3 - 1)}{[20\omega - \phi\{15b_1 + 2\phi(6b_2 + 5\phi b_3)\}](\kappa_1 - \kappa_3)}} \times F\left(\sin^{-1}\left(\sqrt{\frac{\phi\kappa_3 - 1}{\phi(\kappa_3 - \kappa_2)}}\right) | \frac{\kappa_2 - \kappa_3}{\kappa_1 - \kappa_3}\right), \quad (47)$$

where κ_j for $1 \le j \le 3$ are the roots of the cubic polynomial in u given by

$$20u^3\omega - 15b_1u^2 - 12b_2u - 10b_3 = 0, \quad (48)$$

and $F(\psi|m)$ is the elliptic integral of the first kind defined as

$$F(\psi|m) = \int_0^\psi \frac{1}{\sqrt{1 - m\sin^2(\theta)}}\, d\theta, \quad (49)$$

whenever

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2} \quad (50)$$

and

$$m\sin^2(\psi) < 1. \quad (51)$$

5.2. Generalized temporal evolution

The governing model, namely the FLE with generalized temporal evolution, is

$$\begin{aligned} &i(q^l)_t + a(|q|^n q^l)_{xx} + \\ &(b_1|q| + b_2|q|^2 + b_3|q|^3)q^l + i\sigma|q|^2(q^l)_x = \\ &\frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l]. \end{aligned} \quad (52)$$

The corresponding real part equation with the assumption (2) gives

$$\begin{aligned} &a(l+n)\phi^n(x)[\phi(x)\phi''(x) + (l+n-1)\{\phi'(x)\}^2] + \\ &b_1\phi^2(x) + b_2\phi^3(x) + b_3\phi^4(x) - l\omega\phi^2(x) = 0. \end{aligned} \quad (53)$$

For integrability of Eq. (53), the value of *n* needs to be made as given by (44), which reduces (52) and (53) to

$$\begin{aligned} &i(q^l)_t + a(|q|q^l)_{xx} + \\ &(b_1|q| + b_2|q|^2 + b_3|q|^3)q^l + i\sigma|q|^2(q^l)_x = \\ &\frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l], \end{aligned} \quad (54)$$

and

$$\begin{aligned} &a(l+1)\phi(x)[\phi(x)\phi''(x) + l\{\phi'(x)\}^2] + b_1\phi^2(x) + \\ &b_2\phi^3(x) + b_3\phi^4(x) - l\omega\phi^2(x) = 0, \end{aligned} \quad (55)$$

respectively. By the aid of the same translational Lie point symmetry, one arrives at the following implicit solution of Eq. (55) in terms of elliptic integral of the first kind:

$$x = \pm\sqrt{\frac{4a(l+1)(\phi - \kappa_1)(\phi - \kappa_2)(\phi - \kappa_3)}{\left(-\frac{2l\omega}{1+2l} + \frac{\phi b_1}{1+l} + \frac{2\phi^2 b_2}{3+2l} + \frac{\phi^3 b_3}{2+l}\right)\kappa_2(\kappa_1 - \kappa_3)}} \times F\left(\sin^{-1}\left(\sqrt{\frac{\phi(\kappa_3 - \kappa_1)}{(\phi - \kappa_1)\kappa_3}}\right) | \frac{(\kappa_1 - \kappa_2)\kappa_3}{\kappa_2(\kappa_1 - \kappa_3)}\right), \quad (56)$$

where κ_j for $1 \le j \le 3$ is any solution of the following cubic polynomial in *u*:

$$\begin{aligned} &(4b_3l^3 + 12b_3l^2 + 11b_3l + 3b_3)u^3 + \\ &(4b_2l^3 + 14b_2l^2 + 14b_2l + 4b_2)u^2 + \\ &(4b_1l^3 + 16b_1l^2 + 19b_1l + 6b_1)u - \\ &4l^4\omega - 18l^3\omega - 26l^2\omega - 12l\omega = 0. \end{aligned} \quad (57)$$

6. Generalized quadratic-cubic-quartic nonlinearity

Here, the SPM takes the following form:

$$F(|q|^2) = b_1|q|^m + b_2|q|^{2m} + b_3|q|^{2m+1}, \quad (58)$$

where the coefficients b_j for $1 \le j \le 3$ are real-valued constants. The governing FLE with linear temporal evolution and generalized temporal evolution are analyzed in the subsequent subsections.

6.1. Linear temporal evolution

Thus, the FLE with linear temporal evolution is given by

$$iq_t + a(|q|^n q)_{xx} + (b_1|q|^m + b_2|q|^{2m} + b_3|q|^{2m+1})q + i\sigma|q|^2 q_x = \frac{i}{3}[(\sigma - 2\mu)(|q|^2 q)_x + 3\mu(|q|^2)_x q]. \quad (59)$$

Using the expression (2), the real part equation takes the following form:

$$a(n+1)\phi^n(x)[n\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_1\phi^{m+2}(x) + b_2\phi^{2m+2}(x) + b_3\phi^{2m+3}(x) - \omega\phi^2(x) = 0. \quad (60)$$

By virtue of the same translational Lie point symmetry, the following implicit solution emerges:

$$x = \pm\int\sqrt{-\frac{B_1}{B_2}}\,d\phi, \quad (61)$$

where

$$B_1 = a(n+1)(n+2)(m+n+2)\times (2m+n+2)(2m+n+3)\phi^{n-2}, \quad (62)$$

and

$$B_2 = 2[(n+2)\phi^m\{(m+n+2)\phi^m\{b_2(2m+n+3) + b_3(2m+n+2)\phi\} + b_1(2m+n+2)(2m+n+3)\} - \omega(m+n+2)(2m+n+2)(2m+n+3)]. \quad (63)$$

A natural constraint that comes out from (61) is

$$B_1 B_2 < 0 \quad (64)$$

which must remain valid for the solution to exist.

6.2. Generalized temporal evolution

The FLE with generalized temporal evolution shapes up as

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^m + b_2|q|^{2m} + b_3|q|^{2m+1})q^l + i\sigma|q|^2(q^l)_x = \frac{i}{l+2}[(l\sigma - 2\mu)(|q|^2 q^l)_x + \mu(l+2)(|q|^2)_x q^l]. \quad (65)$$

By virtue of (2), the real part equation (9) gives

$$a(l+n)\phi^n(x)[\phi(x)\phi''(x) + (l+n-1)\{\phi'(x)\}^2] + b_1\phi^{m+2}(x) + b_2\phi^{2m+2}(x) + b_3\phi^{2m+3}(x) - -l\omega\phi^2(x) = 0. \quad (66)$$

Using the same translational single Lie point symmetry one arrives at the following solution in quadratures:

$$x = \pm\int\sqrt{\frac{a(l+n)\phi^{n-2}}{2\left\{\frac{l\omega}{2l+n} - \frac{\phi^m b_1}{2l+m+n} - \phi^{2m}\left(\frac{b_2}{2(l+m)+n} + \frac{\phi b_3}{1+2l+2m+n}\right)\right\}}}\,d\phi. \quad (67)$$

7. Conclusions

The current paper recovered quiescent optical solitons that stem from FLE having nonlinear CD and a couple of SPM structures. The Lie symmetry analysis gave way to such results. The solutions are expressed in terms of Appell hypergeometric function, elliptic integral, as well as in quadratures. The most compelling observation is that the model must be free from intermodal dispersion for quiescent solitons to exist. This is true for both forms of SPM structures along with their respective generalized versions. The obtained results are interesting and hopeful for future research. In the following, the governing model will be taken up with differential group delay as well as with dispersion-flattened fibers, which would provide new interesting results. Moreover, additional models such as *e.g.* Radhakrishnan–Kundu–Lakshmanan equation, Lakshmanan–Porsezian–Daniel model, Sasa–Satsuma equation and several others, are yet to be studied. The results of such research activities would be disseminated across, once available after they are aligned and connected with the pre-existing ones [12–15].

Disclosure

The authors claim there is no conflict of interest.

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Неявні стаціонарні оптичні солітони для збуреного рівняння Фокаса–Ленелля з нелінійною хроматичною дисперсією та парою структур з самофазовою модуляцію за симетрією Лі

A.R. Adem, A. Biswas & Y. Yildirim

Анотація. У статті отримано неявні стаціонарні оптичні солітони до збуреного рівняння Фокаса–Ленелля, яке розглядається з нелінійною хроматичною дисперсією та парою структур з самофазовою модуляцією. Вони являють собою квадратично-кубічну та квадратично-кубічно-четвертого степеня форми разом із відповідними узагальненими аналогами. Представлено результати щодо форматів лінійної та узагальненої еволюції в часі. Як інструмент інтегрування у даній роботі використано аналіз симетрії Лі.

Ключові слова: оптичні солітони, симетрія Лі, гіпергеометрична функція Аппелля, еліптичний інтеграл, квадратура.