Optics

Embedded solitons with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities

Y. Yildirim[1], A. Biswas[2,3,4,5], S. Khan[2], M.R. Belic[6]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Near East University, 99138 Nicosia, Cyprus*
[2]*Department of Physics, Chemistry and Mathematics, Alabama A&M University, Normal, AL 35762–4900, USA*
[3]*Department of Mathematics, King Abdulaziz University, Jeddah–21589, Saudi Arabia*
[4]*Department of Applied Mathematics, National Research Nuclear University, 31 Kashirskoe Hwy, Moscow–115409, Russian Federation*
[5]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa–0204, South Africa*
[6]*Science Program, Texas A&M University at Qatar, PO Box 23874, Doha, Qatar*

Abstract. Studied in this work are embedded solitons with quadratic nonlinearity that includes the effect of spatio-temporal dispersion. Two integration schemes yield bright, dark, singular and combo singular soliton solutions from the continuous regime. The existence criteria for these solitons are also included.

Keywords: $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities, embedded solitons.

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1. Introduction

The study of optical solitons in discrete spectrum is quite widespread. There are several results that are reported in its avenue. However, quite less attention has been paid to soliton studies that stem from the continuous regime. There are a few results that have been reported in this context [1–10]. This paper revisits the study of embedded solitons with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities. There are two efficient integration schemes that are implemented in today's work to recover soliton solutions to the model. These give away to bright, dark, singular and combo singular forms of embedded solitons. The existence criteria for these solitons are also listed. The details of the integration procedures along with the spectrum of soliton solutions are all enlisted in the rest of the paper, but first, the governing model with its physical interpretation is illustrated.

1.1. Governing model

The governing model with the quadratic nonlinearity [1–10] is as follows:

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 q^* r + d_1 |q|^2 q = 0\,, \tag{1}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + d_2 q^2 + \delta |q|^2 r = 0. \tag{2}$$

In this model, the independent variables x and t represent the spatial and temporal variables, respectively. The constants a_j provide the chromatic dispersion, while the constants b_j assure the existence of spatio-temporal dispersion (STD). Also, the coefficients c_j provide the existence of group-velocity mismatch because of frequency difference between fundamental harmonics (FH) and second harmonics (SH) fields that are given by the complex valued functions $q(x, t)$, $r(x, t)$, respectively. This model specifically governs embedded solitons that are nonlinear waves and become confined to the continuous spectrum of a nonlinear system. These solitons arise in presence of opposing dispersion and competing nonlinearities at FH and SH.

2. Mathematical analysis

To start off, the basic assumptions are

$$q(x,t) = P_1(\zeta) e^{i\varphi(x,t)}\,, \tag{3}$$

$$r(x,t) = P_2(\zeta) e^{2i\varphi(x,t)}\,, \tag{4}$$

where

$$\zeta = \eta\left(x - vt\right), \tag{5}$$

and v stands for the speed of the wave. Next, the phase φ is structured as

$$\varphi(x,t) = -\kappa x + \omega t + \theta_0, \tag{6}$$

where the frequency, wave number and phase constant are designated as κ, ω and θ_0, respectively. Insert (3) and (4) into (1) and (2). Then, from (1), real and imaginary parts fall out

$$\eta^2(b_1 v - a_1)P_1'' + \left(\omega + a_1\kappa^2 - b_1\omega\kappa\right)P_1 - c_1 P_1 P_2 - d_1 P_1^3 = 0, \tag{7}$$

$$v = \frac{b_1\omega - 2a_1\kappa}{1 - b_1\kappa}, \tag{8}$$

respectively. And then from (2)

$$\eta^2(b_2 v - a_2)P_2'' + \left(2\omega + 4a_2\kappa^2 - 4b_2\omega\kappa - c_2\right)P_2 - \\ - d_2 P_1^2 - \delta P_1^2 P_2 = 0, \tag{9}$$

$$v = \frac{2b_2\omega - 4a_2\kappa}{1 - 2b_2 k}, \tag{10}$$

Eqs (7) – (10) reduce to

$$2\eta^2(bv - a)P_1'' + \left(\omega + 2a\kappa^2 - 2b\omega\kappa\right)P_1 - \\ - c_1 P_1 P_2 - d_1 P_1^3 = 0, \tag{11}$$

$$\eta^2(bv - a)P_2'' + \left(2\omega + 4a\kappa^2 - 4b\omega\kappa - c_2\right)P_2 - \\ - d_2 P_1^2 - \delta P_1^2 P_2 = 0, \tag{12}$$

$$v = \frac{2b\omega - 4a\kappa}{1 - 2bk}, \tag{13}$$

as long as

$$a_1 = 2a, \ a_2 = a, \ b_1 = 2b, \ b_2 = b. \tag{14}$$

Thus, the governing equations can be written as

$$iq_t + 2aq_{xx} + 2bq_{xt} + c_1 q^* r + d_1|q|^2 q = 0, \tag{15}$$

$$ir_t + ar_{xx} + br_{xt} + c_2 r + d_2 q^2 + \delta|q|^2 r = 0. \tag{16}$$

Eqs (11) and (12) are employed to retrieve solitons for Eqs (15) and (16).

2.1. F-expansion procedure

The solution structure of (11) and (12) is considered to be

$$P_1(\zeta) = \sum_{i=0}^{N} A_i F^i(\zeta), \tag{17}$$

$$P_2(\zeta) = \sum_{i=0}^{N} B_i F^i(\zeta), \tag{18}$$

where A_i and B_i for $1 \le i \le N$ are constants that need to be designated, and the number N originates from balancing principle. The function $F(\zeta)$ obeys the form:

$$F'(\zeta) = \sqrt{PF^2(\zeta) + QF(\zeta) + R}, \tag{19}$$

where P, Q and R are constants. It is necessary to note that the solutions of (19) are as follows:

$$\begin{aligned}
&F(\zeta) = \mathrm{sn}(\zeta) = \tanh(\zeta), \ P = m^2, \\
&Q = -(1 + m^2), \ R = 1, \ m \to 1, \\
&F(\zeta) = \mathrm{ns}(\zeta) = \coth(\zeta), \ P = 1, \\
&Q = -(1 + m^2), \ R = m^2, \ m \to 1, \\
&F(\zeta) = \mathrm{sc}(\zeta) = \tan(\zeta), \ P = 1 - m^2, \\
&Q = 2 - m^2, \ R = 1, \ m \to 0, \\
&F(\zeta) = \mathrm{cs}(\zeta) = \cot(\zeta), \ P = 1, \\
&Q = 2 - m^2, \ R = 1 - m^2, \ m \to 0, \\
&F(\zeta) = \mathrm{cn}(\zeta) = \mathrm{sech}(\zeta), \ P = -m^2, \\
&Q = 2m^2 - 1, \ R = 1 - m^2, \ m \to 1, \\
&F(\zeta) = \mathrm{ds}(\zeta) = \mathrm{csch}(\zeta), \ P = 1, \\
&Q = 2m^2 - 1, \ R = -m^2\left(1 - m^2\right), \ m \to 1, \\
&F(\zeta) = \mathrm{nc}(\zeta) = \sec(\zeta), \ P = 1 - m^2, \\
&Q = 2m^2 - 1, \ R = -m^2, \ m \to 0, \\
&F(\zeta) = \mathrm{ns}(\zeta) = \csc(\zeta), \ P = 1, \\
&Q = -(1 + m^2), \ R = m^2, \ m \to 0, \\
&F(\zeta) = \mathrm{ns}(\zeta) \pm \mathrm{ds}(\zeta) = \coth(\zeta) \pm \mathrm{csch}(\zeta), \ P = \frac{1}{4}, \\
&Q = \frac{m^2 - 2}{2}, \ R = \frac{m^2}{4}, \ m \to 1, \\
&F(\zeta) = \mathrm{sn}(\zeta) \pm i\,\mathrm{cn}(\zeta) = \tanh(\zeta) \pm i\,\mathrm{sech}(\zeta), \ P = \frac{m^2}{4}, \\
&Q = \frac{m^2 - 2}{2}, \ R = \frac{m^2}{4}, \ m \to 1, \\
&F(\zeta) = \mathrm{ns}(\zeta) \pm \mathrm{cs}(\zeta) = \csc(\zeta) \pm \cot(\zeta), \ P = \frac{1}{4}, \\
&Q = \frac{1 - 2m^2}{2}, \ R = \frac{1}{4}, \ m \to 0, \\
&F(\zeta) = \mathrm{nc}(\zeta) \pm \mathrm{sc}(\zeta) = \sec(\zeta) \pm \tan(\zeta), \ P = \frac{1 - m^2}{4}, \\
&Q = \frac{1 + m^2}{2}, \ R = \frac{1 - m^2}{4}, \ m \to 0.
\end{aligned} \tag{20}$$

From the balancing principle, (17) and (18) take the form:

$$P_1(\zeta) = A_0 + A_1 F(\zeta), \tag{21}$$

$$P_2(\zeta) = B_0 + B_1 F(\zeta) + B_2 F^2(\zeta). \tag{22}$$

Plugging (21) and (22) along with (19) into (11) and (12) results in

$$A_0 = 0,\ B_1 = 0,\ B_2 = \frac{\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)QB_0}{\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)R},$$

$$\eta = \pm\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{2Q(-bv + a)}},\ A_1 = \pm\sqrt{-\frac{3P\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{Q\delta}},$$

$$d_1 = \frac{\delta\begin{pmatrix} 8PRa^2\kappa^4 - 16PRab\kappa^3\omega - 8PRa\kappa^2B_0c_1 + 4Q^2a\kappa^2B_0c_1 \\ + 8PRa\kappa^2\omega + 8PRb^2\kappa^2\omega^2 + 8PRb\kappa\omega B_0c_1 - 4Q^2b\kappa\omega B_0c_1 \\ - 8PRb\kappa\omega^2 + 2PRB_0{}^2c_1{}^2 - 4PR\omega B_0c_1 + 2Q^2\omega B_0c_1 - Q^2B_0c_1c_2 + 2PR\omega^2 \end{pmatrix}}{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)^2 PR},$$

$$d_2 = -\frac{\delta B_0\begin{pmatrix} Q^2c_2{}^2 - 2Q^2\omega c_2 + 3PR\omega^2 - 2Q^2B_0c_1c_2 + 4Q^2\omega B_0c_1 - 6PR\omega B_0c_1 \\ + 4Q^2b\kappa\omega c_2 - 4Q^2a\kappa^2c_2 + 3PRB_0{}^2c_1{}^2 - 12PRb\kappa\omega^2 + 12PRa\kappa^2\omega \\ - 8Q^2b\kappa\omega B_0c_1 + 8Q^2a\kappa^2B_0c_1 + 12PRb\kappa\omega B_0c_1 - 12PRa\kappa^2B_0c_1 \\ + 12PRb^2\kappa^2\omega^2 - 24PRab\kappa^3\omega + 12PRa^2\kappa^4 \end{pmatrix}}{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)^2 PR}. \tag{23}$$

Inserting (23) into (21) and (22) yields dark solitons

$$q(x,t) = \pm\sqrt{\frac{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{2\delta}}\tanh\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4(bv - a)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]e^{i(-\kappa x + \omega t + \theta_0)}, \tag{24}$$

$$r(x,t) = \left\{B_0 - \frac{2\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)B_0}{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}\tanh^2\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4(bv - a)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]\right\}e^{2i(-\kappa x + \omega t + \theta_0)}, \tag{25}$$

singular solitons

$$q(x,t) = \pm\sqrt{\frac{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{2\delta}}\coth\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4(bv - a)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]e^{i(-\kappa x + \omega t + \theta_0)}, \tag{26}$$

$$r(x,t) = \left\{B_0 - \frac{2\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)B_0}{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}\coth^2\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4(bv - a)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]\right\}e^{2i(-\kappa x + \omega t + \theta_0)}, \tag{27}$$

and combo singular solitons

$$q(x,t) = \pm\sqrt{\frac{6\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{4\delta}}\begin{pmatrix} \coth\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{bv - a}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right] \\ \pm\operatorname{csch}\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{bv - a}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right] \end{pmatrix}^2 e^{i(-\kappa x + \omega t + \theta_0)}, \tag{28}$$

$$r(x,t)=\left\{\begin{array}{l} B_0-\dfrac{2\left(4a\kappa^2-4b\kappa\omega+2\omega-c_2\right)B_0}{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega} \\ \times\left(\begin{array}{l}\coth\left[\sqrt{\dfrac{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega}{bv-a}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right] \\ \pm\operatorname{csch}\left[\sqrt{\dfrac{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega}{bv-a}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{array}\right)^2\end{array}\right\}e^{2i\left(-\kappa x+\omega t+\theta_0\right)}. \tag{29}$$

These soliton solutions are valid for

$$\delta\left(2a\kappa^2-2b\kappa\omega-B_0c_1+\omega\right)>0, \tag{30}$$

$$(bv-a)\left(2a\kappa^2-2b\kappa\omega-B_0c_1+\omega\right)>0. \tag{31}$$

2.2. Sine–Gordon equation approach

The solution structures of (11) and (12) according to this approach are taken to be

$$P_1(\zeta)=\sum_{i=1}^{N}\cos^{i-1}(V(\zeta))\left[B_i\sin(V(\zeta))+A_i\cos(V(\zeta))\right]+A_0, \tag{32}$$

$$P_2(\zeta)=\sum_{i=1}^{N}\cos^{i-1}(V(\zeta))\left[D_i\sin(V(\zeta))+C_i\cos(V(\zeta))\right]+C_0, \tag{33}$$

where A_i, B_i, C_i, and D_i for $1\le i\le N$ are constants, the number N is determined from balancing principle, and $V(\zeta)$ holds

$$V'(\zeta)=\sin(V(\zeta)). \tag{34}$$

Also, it needs to be mentioned that (34) has the following solutions:

$$\sin(V(\zeta))=\operatorname{sech}(\zeta)\ \text{ or }\ \sin(V(\zeta))=\operatorname{icsh}(\zeta),\ \cos(V(\zeta))=\tanh(\zeta)\ \text{ or }\ \cos(V(\zeta))=\coth(\zeta). \tag{35}$$

The balancing principle implies that

$$P_1(\zeta)=B_1\sin(V(\zeta))+A_1\cos(V(\zeta))+A_0, \tag{36}$$

$$P_2(\zeta)=\cos(V(\zeta))\left[D_2\sin(V(\zeta))+C_2\cos(V(\zeta))\right]+D_1\sin(V(\zeta))+C_1\cos(V(\zeta))+C_0. \tag{37}$$

Inserting (36) and (37) along with (34) into (11) and (12) leads to

$$\eta=\pm\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2(\delta-6d_1)(-bv+a)}},\ A_0=0,\ A_1=0,$$

$$B_1=\pm\sqrt{-\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta(\delta-6d_1)}},\ D_1=0,\ C_1=0,\ D_2=0,$$

$$C_0=-\frac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta(\delta-6d_1)c_1},$$

$$C_2=\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)(2\delta-3d_1)}{\delta(\delta-6d_1)c_1},\ c_2=4a\kappa^2-4b\kappa\omega+2\omega. \tag{38}$$

Substituting (38) along with (35) in (36) and (37) yields soliton solutions

$$q(x,t)=\pm\sqrt{-\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta\left(\delta-6d_1\right)}}\operatorname{sech}\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{39}$$

$$r(x,t)=\left\{\begin{aligned}&-\frac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta\left(\delta-6d_1\right)c_1}\\&+\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)\left(2\delta-3d_1\right)}{\delta\left(\delta-6d_1\right)c_1}\\&\times\tanh^2\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{aligned}\right\}e^{2i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{40}$$

$$q(x,t)=\pm\sqrt{\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta\left(\delta-6d_1\right)}}\operatorname{csch}\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{41}$$

$$r(x,t)=\left\{\begin{aligned}&-\frac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta\left(\delta-6d_1\right)c_1}+\\&+\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)\left(2\delta-3d_1\right)}{\delta\left(\delta-6d_1\right)c_1}\times\\&\times\coth^2\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{aligned}\right\}e^{2i\left(-\kappa x+\omega t+\theta_0\right)}. \tag{42}$$

Dark soliton (40) and singular soliton (42) are valid for

$$\left(\delta-6d_1\right)\left(-bv+a\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)<0, \tag{43}$$

while bright soliton (39) is valid for the constraint (43) along with

$$\delta\left(\delta-6d_1\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)<0 \tag{44}$$

and singular soliton (41) is valid for the constraint (43) along with

$$\delta\left(\delta-6d_1\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)>0. \tag{45}$$

3. Conclusions

Retrieved in this paper have been bright, dark, singular and combo singular embedded optical soliton solutions with the quadratic nonlinearity. A couple of integration schemes have been implemented to make this retrieval possible. The soliton solutions appeared with their respective existence criteria. Thus, the obtained results have paved its way to further future developments. One can handle embedded solitons using the variational principle as a sequel to previously reported results [5]. This time it needs to be studied using additional pulse formats. Another avenue to explore in this area is to handle the problem using the Lie symmetry analysis that will be a continuation and extension to previously recovered results [3]. These studies are under way, and their results will be soon published.

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Authors and CV

Yakup Yildirim received the PhD degree in Mathematics from Uludag University, Turkey, in 2019. He is Assistant Professor at Near East University, Cyprus. The area of scientific interests includes optical soliton solutions and conservation laws. He is the author of more than 100 publications.

Anjan Biswas earned his MA and PhD degrees from the University of New Mexico in Albuquerque, NM, USA. Subsequently, he carried out his Post-Doctoral studies at the University of Colorado, Boulder, CO, USA. Currently, he works as a faculty member at Alabama A&M University, Huntsville, AL. His research focus is on mathematical photonics.

Salam Khan earned his PhD degree in Applied Mathematics from the University of Electrocommunications, Tokyo, Japan. Subsequently, he carried out his Post-Doctoral studies at Florida State University, Tallahassee, FL, USA. Currently he is an Associate Professor of Mathematics at Alabama A&M University, Huntsville, AL, USA. His research areas include applied mathematics, statistics, stochastic theory and mathematical physics.

Milivoj R. Belic, PhD, Al Sraiya Holding Professor in physics at the Texas A & M University at Qatar. Since 1982, he is affiliated with the Institute of Physics, Belgrade in Serbia. His research areas include nonlinear optics and nonlinear dynamics. He is the author of 6 books and more than 600 papers that attracted more than 10,000 citations; his *h*-index is 46, according to Google Scholar. The recipient of numerous research awards, Dr. Belic received the Galileo Galilei Award for 2004, from the International Commission for Optics, and the Research Team Award from the Qatar National Research Fund in 2012 and 2014. He is Senior Member of the Optical Society of America.

Вбудовані солітони з $\chi^{(2)}$ і $\chi^{(3)}$ нелінійною сприйнятливістю

Y. Yildirim, A. Biswas, S. Khan, M.R. Belic

Анотація. У цій роботі досліджено вбудовані солітони з квадратичною нелінійністю з урахуванням ефекту просторово-часової дисперсії. Обидві схеми інтегрування приводять до отримання яскравих, темних, сингулярних та комбінованих сингулярних солітонних розв’язків у неперервному режимі. Враховано також критерії існування цих солітонів.

Ключові слова: $\chi^{(2)}$ і $\chi^{(3)}$ нелінійності, вбудовані солітони.