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Highly dispersive optical solitons with differential group delay for the Kerr law of self-phase modulation

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Abstract. The paper addresses highly dispersive optical solitons with differential group delay, characterized by the Kerr law of self-phase modulation. The extended auxiliary equation approach recovers the full spectrum of solitons. Additionally, solutions in terms of Weierstrass's elliptic functions are reported. The results are all presented with the necessary parameter constraints that naturally emerge from the scheme.

Keywords: solitons, dispersion, birefringence.

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1. Introduction

The concept of highly dispersive optical solitons was conceived slightly less than a decade ago. This idea came on board when the chromatic dispersion (CD) runs low which is the main source of dispersive effects for the propagation of solitons through optical fibers. To compensate for this low count, additional dispersive effects were included so that the relation between CD and self-phase modulation (SPM) effect stays balanced. These dispersive effects stem from intermodal dispersion (IMD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD). It must be noted that these higher-order dispersions naturally have their shortcomings, namely the soliton radiation as well as the drastic slow-down of solitons. However, these effects are neglected by default.

It is now time to move on to the next chapter. Therefore, upon turning the page, the concept of highly dispersive effect is taken up in birefringent fibers. The model for these solitons is addressed with the Kerr law of SPM that automatically kicks in the so-called cross-phase modulation (XPM) effect. It must be noted that the effect of four-wave mixing is also neglected in this extended version of the Manakov equation. Additionally, the study of highly dispersive optical solitons for birefringent fibers was already carried out and its soliton solutions together with the conservation laws were all reported [1]. The current paper, however, revisits this concept in birefringent fibers with the Kerr law of SPM. The integrability scheme here is the extended auxiliary equation approach. This comprehensive scheme reveals a full spectrum of optical solitons along with additional solutions that are in terms of the Weierstrass elliptic functions. The details are exhibited in the rest of the paper along with the parameter constraints, that are necessary for the existence of these solitons, after a succinct introduction to the model.

1.1. Mathematical model

The dimensionless form of the nonlinear Schrödinger's equation for highly dispersive optical solitons in birefringent fibers with an account of Kerr's law of nonlinear refractive index can be written as [1]:

$$iu_t + ia_1^1 u_x + a_2^1 u_{xx} + ia_3^1 u_{xxx} + a_4^1 u_{xxxx} + ia_5^1 u_{xxxxx} + a_6^1 u_{xxxxxx} + (b_1^1|u|^2 + b_2^1|v|^2)u = 0, \quad (1)$$

$$iv_t + ia_1^2 v_x + a_2^2 v_{xx} + ia_3^2 v_{xxx} + a_4^2 v_{xxxx} + ia_5^2 v_{xxxxx} + a_6^2 v_{xxxxxx} + (b_1^2|v|^2 + b_2^2|u|^2)v = 0, \quad (2)$$

where the real-valued coefficients a_j^l for $1 \le j \le 6$ represent IMD, CD, 3OD, 4OD, 5OD, and 6OD, respectively, along these two components of birefringent fibers for $l = 1, 2$. Also, b_l^l ($l = 1, 2$) represents SPM and XPM.

2. Preliminary mathematical analysis

In this section, we will suppose that Eqs. (2) and (3) possess the following solutions [1–10]:

$$u(x,t) = H_1(\zeta)\exp[iH(x,t)], \quad v(x,t) = H_2(\zeta)\exp[iH(x,t)], \quad (3)$$

and

$$\zeta = x - ct, \quad H(z,t) = -\kappa x + \Omega t + \varsigma_0. \quad (4)$$

Assuming that c, κ, Ω and ς_0 are all non-zero parameters, where c represents the soliton velocity, κ denotes its wave number, Ω represents its frequency, and ς_0 is the phase constant, we have real functions $H_1(\zeta), H_2(\zeta)$, and $H(x,t)$ that represent the amplitude and phase components of the soliton, respectively. Eqs. (1) and (2) may be changed to Eqs. (5) and (6) by isolating their real and imaginary parts. From this, we can conclude that:

$$\Re_1: a_6^1 H_1^{(6)}(\zeta) + (a_4^1 + 5a_5^1\kappa - 15a_6^1\kappa^2)H_1^{(4)}(\zeta) + (a_2^1 + 3a_3^1\kappa - 6a_4^1\kappa^2 - 10a_5^1\kappa^3 + 15a_6^1\kappa^4)H_1''(\zeta) - -(\Omega - a_1^1\kappa + a_2^1\kappa^2 + a_3^1\kappa^3 - a_4^1\kappa^4 - a_5^1\kappa^5 + a_6^1\kappa^6) \times H_1(\zeta) + b_1^1 H_1^3(\zeta) + b_2^1 H_1(\zeta)H_2^2(\zeta) = 0, \quad (5)$$

$$\Re_2: a_6^2 H_2^{(6)}(\zeta) + (a_4^2 + 5a_5^2\kappa - 15a_6^2\kappa^2)H_2^{(4)}(\zeta) + (a_2^2 + 3a_3^2\kappa - 6a_4^2\kappa^2 - 10a_5^2\kappa^3 + 15a_6^2\kappa^4)H_2''(\zeta) - -(\Omega - a_1^2\kappa + a_2^2\kappa^2 + a_3^2\kappa^3 - a_4^2\kappa^4 - a_5^2\kappa^5 + a_6^2\kappa^6) \times H_2(\zeta) + b_1^2 H_2^3(\zeta) + b_2^2 H_2(\zeta)H_1^2(\zeta) = 0, \quad (6)$$

and

$$\Im_1: (a_5^1 - 6a_6^1\kappa)H_1^{(5)}(\zeta) + (a_3^1 - 4a_4^1\kappa - 10a_5^1\kappa^2 + 20a_6^1\kappa^3)H_1'''(\zeta) -(c - a_1^1 + 2a_2^1\kappa + 3a_3^1\kappa^2 - 4a_4^1\kappa^3 - -5a_5^1\kappa^4 + 6a_6^1\kappa^5)H_1'(\zeta) = 0, \quad (7)$$

$$\Im_2: (a_5^2 - 6a_6^2\kappa)H_2^{(5)}(\zeta) + (a_3^2 - 4a_4^2\kappa - 10a_5^2\kappa^2 + 20a_6^2\kappa^3)H_2'''(\zeta) -(c - a_1^2 + 2a_2^2\kappa + 3a_3^2\kappa^2 - 4a_4^2\kappa^3 - -5a_5^2\kappa^4 + 6a_6^2\kappa^5)H_2'(\zeta) = 0. \quad (8)$$

Set

$$H_2(\zeta) = AH_1(\zeta), \quad (9)$$

is provided for $A \neq 0, 1$. Now, Eqs. (5)–(8) become

$$\Re_1: a_6^1 H_1^{(6)}(\zeta) + (a_4^1 + 5a_5^1\kappa - 15a_6^1\kappa^2)H_1^{(4)}(\zeta) + (a_2^1 + 3a_3^1\kappa - 6a_4^1\kappa^2 - 10a_5^1\kappa^3 + 15a_6^1\kappa^4)H_1''(\zeta) -(\Omega - a_1^1\kappa + a_2^1\kappa^2 + a_3^1\kappa^3 - a_4^1\kappa^4 - -a_5^1\kappa^5 + a_6^1\kappa^6)H_1(\zeta) + (b_1^1 + A^2 b_2^1)H_1^3(\zeta) = 0, \quad (10)$$

$$\Re_2: a_6^2 H_1^{(6)}(\zeta) + (a_4^2 + 5a_5^2\kappa - 15a_6^2\kappa^2)H_1^{(4)}(\zeta) + (a_2^2 + 3a_3^2\kappa - 6a_4^2\kappa^2 - 10a_5^2\kappa^3 + 15a_6^2\kappa^4)H_1''(\zeta) + -(\Omega - a_1^2\kappa + a_2^2\kappa^2 + a_3^2\kappa^3 - a_4^2\kappa^4 - a_5^2\kappa^5 + +a_6^2\kappa^6)H_1(\zeta) + (b_1^2 A^2 + b_2^2)H_1^3(\zeta) = 0, \quad (11)$$

$$\Im_1: (a_5^1 - 6a_6^1\kappa)H_1^{(5)}(\zeta) + (a_3^1 - 4a_4^1\kappa - 10a_5^1\kappa^2 + 20a_6^1\kappa^3)H_1'''(\zeta) -(c - a_1^1 + 2a_2^1\kappa + 3a_3^1\kappa^2 - 4a_4^1\kappa^3 - -5a_5^1\kappa^4 + 6a_6^1\kappa^5)H_1'(\zeta) = 0, \quad (12)$$

$$\Im_2: (a_5^2 - 6a_6^2\kappa)H_1^{(5)}(\zeta) + (a_3^2 - 4a_4^2\kappa - 10a_5^2\kappa^2 + 20a_6^2\kappa^3)H_1'''(\zeta) + -(c - a_1^2 + 2a_2^2\kappa + 3a_3^2\kappa^2 - 4a_4^2\kappa^3 - -5a_5^2\kappa^4 + 6a_6^2\kappa^5)H_1'(\zeta) = 0. \quad (13)$$

By equating the coefficients of the linearly independent functions in equations (12) and (13) to zero, we obtain:

$$\kappa = \frac{a_5^l}{6a_6^l}, l = 1, 2, \quad (14)$$

$$c = a_1^l - 2a_2^l\kappa - 3a_3^l\kappa^2 + 4a_4^l\kappa^3 + +5a_5^l\kappa^4 - 6a_6^l\kappa^5, \qquad l = 1, 2, \quad (15)$$

$$a_3^l - 4a_4^l\kappa - 10a_5^l\kappa^2 + 20a_6^l\kappa^3 = 0, \qquad l = 1, 2. \quad (16)$$

Equations (10) and (11) exhibit identical forms, under the following constraint conditions:

$$\left.\begin{aligned} & a_6^1 = a_6^2, \\ & a_4^1 + 5a_5^1\kappa - 15a_6^1\kappa^2 = a_4^2 + 5a_5^2\kappa - 15a_6^2\kappa^2, \\ & a_2^1 + 3a_3^1\kappa - 6a_4^1\kappa^2 - 10a_5^1\kappa^3 + 15a_6^1\kappa^4 = \\ & = a_2^2 + 3a_3^2\kappa - 6a_4^2\kappa^2 - 10a_5^2\kappa^3 + 15a_6^2\kappa^4, \\ & \Omega - a_1^1\kappa + a_2^1\kappa^2 + a_3^1\kappa^3 - a_4^1\kappa^4 - a_5^1\kappa^5 + a_6^1\kappa^6 = \\ & = \Omega - a_1^2\kappa + a_2^2\kappa^2 + a_3^2\kappa^3 - a_4^2\kappa^4 - a_5^2\kappa^5 + a_6^2\kappa^6, \\ & b_1^1 + A^2 b_2^1 = b_1^2 A^2 + b_2^2. \end{aligned}\right\} \quad (17)$$

From (14)–(17), one can derives the following:

$$\left.\begin{array}{l} a_6^1 = \dfrac{6a_6^1 a_4^2 - 5(a_5^1 - a_5^2)a_5^1}{6a_4^1}, \\ a_2^1 = a_2^2 - 3(a_3^1 - a_3^2)\kappa - 6(a_4^2 - a_4^1)\kappa^2 + \\ \quad +10(a_5^1 - a_5^2)\kappa^3 - 15(a_6^1 - a_6^2)\kappa^4, \\ a_4^2 = \\ \dfrac{(a_1^2 - a_1^1)\kappa - 2(a_3^1 - a_3^2)\kappa^3 + 5a_4^1\kappa^4 + 9(a_5^1 - a_5^2)\kappa^5}{5\kappa^4}, \\ b_1^1 = (b_1^2 - b_2^1)A^2 + b_2^2. \end{array}\right\} \tag{18}$$

We can express equation (10) in an alternative form as follows:

$$H_1^{(6)}(\zeta) + \Delta_1 H_1^{(4)}(\zeta) + \Delta_2 H_1''(\zeta) - \Delta_3 H_1(\zeta) + +\Delta_4 H_1^3(\zeta) = 0, \tag{19}$$

where

$$\left.\begin{array}{l} \Delta_1 = \dfrac{(a_4^1 + 5a_5^1\kappa - 15a_6^1\kappa^2)}{a_6^1}, \\ \Delta_2 = \dfrac{(a_2^1 + 3a_3^1\kappa - 6a_4^1\kappa^2 - 10a_5^1\kappa^3 + 15a_6^1\kappa^4)}{a_6^1}, \\ \Delta_3 = \dfrac{(\Omega - a_1^1\kappa + a_2^1\kappa^2 + a_3^1\kappa^3 - a_4^1\kappa^4 - a_5^1\kappa^5 + a_6^1\kappa^6)}{a_6^1}, \\ \Delta_4 = \dfrac{(b_1^1 + A^2 b_2^1)}{a_6^1}. \end{array}\right\} \tag{20}$$

Next, we will solve Eq. (19) using the following method.

3. Extended auxiliary equation approach

This method assumes the existence of a formal solution to Eq. (19):

$$H_1(\zeta) = \sum_{l=0}^{N} E_l f^l(\zeta), \tag{21}$$

where $f(\zeta)$ satisfies:

$$f'^2(\zeta) = \sum_{j=0}^{4} h_j f^j(\zeta). \tag{22}$$

Here, E_l and h_j are constants, with $E_N \neq 0$ and $h_4 \neq 0$, where $N \in \mathbb{Z}^+$. It is commonly known that Eq. (22) has the following solutions:

Set–1: If we set $h_0 = h_1 = h_3 = 0$, then Eq. (22) has the solutions:

(I): Bright solitons:

$$f(\zeta) = \pm\sqrt{-\frac{h_2}{h_4}}\, sech\left(\sqrt{h_2}\zeta\right), \quad h_2 > 0, h_4 < 0. \tag{23}$$

(II): Singular solitons:

$$f(\zeta) = \pm\sqrt{\frac{h_2}{h_4}}\, csch\left(\sqrt{h_2}\zeta\right), \quad h_2 > 0, h_4 > 0. \tag{24}$$

Set–2: If we set $h_1 = h_3 = 0$ and $h_0 = \frac{h_2^2}{4h_4}$, then Eq. (22) has the solutions:

(I): Dark solitons:

$$f(\zeta) = \pm\sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\zeta\right), \quad h_2 < 0, h_4 > 0. \tag{25}$$

(II): Singular solitons:

$$f(\zeta) = \pm\sqrt{-\frac{h_2}{2h_4}}\coth\left(\sqrt{-\frac{h_2}{2}}\zeta\right), \quad h_2 < 0, h_4 > 0. \tag{26}$$

Set–3: If we set $h_0 = h_1 = 0$, then Eq. (22) has the combo-bright-singular soliton solutions:

$$f(\zeta) = \frac{2h_2}{\pm 2\sqrt{h_2 h_4}\sinh\left(\sqrt{h_2}\zeta\right) + h_3\left[\cosh\left(\sqrt{h_2}\zeta\right) - 1\right]}, \quad h_2 < 0, h_4 > 0. \tag{27}$$

Set–4: If we set $h_0 = h_1 = 0$ and $h_3 = 2\sqrt{h_2 h_4}$, then Eq. (22) has the solutions:

(I): Dark solitons:

$$f(\zeta) = -\frac{1}{2}\sqrt{\frac{h_2}{h_4}}\left[1 \pm \tanh\left(\frac{1}{2}\sqrt{h_2}\zeta\right)\right], \quad h_2 > 0, h_4 > 0. \tag{28}$$

(II): Singular solitons:

$$f(\zeta) = -\frac{1}{2}\sqrt{\frac{h_2}{h_4}}\left[1 \pm \coth\left(\frac{1}{2}\sqrt{h_2}\zeta\right)\right], \quad h_2 > 0, h_4 > 0. \tag{29}$$

Set–5: If we set $h_1 = h_3 = 0$, then Eq. (22) yields the following Weierstrass elliptic function solutions:

$$f(\zeta) = \frac{3\wp'\left(\zeta, h_0 h_4 + \frac{h_2}{12}, \frac{h_2(36h_0h_4 - h_2^2)}{216}\right)}{\sqrt{h_4}\left[6\wp\left(\zeta, h_0 h_4 + \frac{h_2}{12}, \frac{h_2(36h_0h_4 - h_2^2)}{216}\right) + h_2\right]}, \quad h_4 > 0, \tag{30}$$

and

$$f(\zeta) = \sqrt{\frac{3\wp\left(\zeta, \frac{4}{3}h_2^2 - 3h_0h_4, \frac{4}{27}(9h_0h_2h_4 - 2h_2^3)\right) - h_2}{3h_4}}. \tag{31}$$

Here, $\wp(\zeta, g_2, g_3)$ represents the Weierstrass elliptic function, and $\wp'(\zeta, g_2, g_3) = \frac{d\wp(\zeta, g_2, g_3)}{d\zeta}$, which satisfies: $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, where g_2 and g_3 are the invariants of the Weierstrass elliptic function.

3.1. Soliton solutions

In Eq. (19), the balance number $N = 3$. Consequently, Eq. (19) has the formal solution:

$$H_1(\zeta) = E_0 + E_1 f(\zeta) + E_2 f^2(\zeta) + E_3 f^3(\zeta), \tag{32}$$

where E_0, E_1, E_2 and E_3 are constants to be determined providing $E_3 \neq 0$.

By substituting (32) and (22) into (19), we derive the following system of equations:

$$
\left.
\begin{aligned}
&\Delta_4 E_3^3 + 20160 h_4^3 E_3 = 0,\\
&5040 h_4^3 E_2 + 41580 h_4^2 E_3 h_3 + 3\Delta_4 E_2 E_3^2 = 0,\\
&360\Delta_1 h_4^2 E_3 + 29880 h_4^2 h_2 E_3 + 3\Delta_4 E_2^2 E_3 + 26730 h_4 E_3 h_3^2 + 720 h_4^3 E_1 + 9720 h_4^2 h_3 E_2 +\\
&\qquad + 3\Delta_4 E_1 E_3^2 = 0,\\
&35910 h_4 h_3 E_3 h_2 + 3\Delta_4 E_0 E_3^2 + 540\Delta_1 h_3 E_3 h_4 + 120\Delta_1 h_4^2 E_2 + 6\Delta_4 E_1 E_2 E_3 + \frac{10395}{2} h_3^3 E_3 +\\
&+\Delta_4 E_2^3 + 6720 h_4^2 E_2 h_2 + 5670 h_4 h_3^2 E_2 + 22680 h_4^2 E_3 h_1 + 1260 h_4^2 E_1 h_3 = 0,\\
&5040 h_4^2 E_2 h_1 + 18144 h_4^2 h_0 E_3 + 408\Delta_1 h_2 E_3 h_4 + 840 h_4^2 E_1 h_2 + 189\Delta_1 h_3^2 E_3 + 12\Delta_2 E_3 h_4\\
&+7140 h_4 h_3 E_2 h_2 + 25704 h_4 h_3 E_3 h_1 + 24\Delta_1 h_4^2 E_1 + 630 h_4 E_1 h_3^2 + 9450 h_3^2 h_2 E_3 +\\
&+6\Delta_4 E_0 E_2 E_3 + 3\Delta_4 E_1 E_2^2 + 168\Delta_1 h_3 E_2 h_4 + 945 h_3^3 E_2 + 11172 h_4 h_2^2 E_3 + 3\Delta_4 E_1^2 E_3 = 0,\\
&144 h_4 E_2 h_0^2 + 270 h_3 h_0^2 E_3 + \frac{9}{4} h_3 h_1^2 E_1 + \frac{1}{2} h_2^2 E_1 h_1 + 32 h_2^2 h_0 E_2 + \frac{15}{2} h_2 h_1^2 E_2 + \frac{3}{2}\Delta_1 h_1^2 E_2\\
&+\frac{1}{2}\Delta_2 E_1 h_1 + 2\Delta_2 h_0 E_2 + 8\Delta_1 h_2 h_0 E_2 + \Delta_4 E_0^3 + 36 h_4 E_1 h_0 h_1 + 15 h_3 E_1 h_0 h_2 + 99 h_3 h_0 E_2 h_1\\
&+225 h_2 h_0 E_3 h_1 + \frac{45}{4} E_3 h_1^3 + 3\Delta_1 E_1 h_0 h_3 + 18\Delta_1 h_0 E_3 h_1 - \Delta_3 E_0 + \frac{1}{2}\Delta_1 h_2 E_1 h_1 = 0,\\
&\frac{21}{2}\Delta_2 E_3 h_3 + 2016 h_4 h_2^2 E_2 + \frac{315}{4} E_1 h_3^3 + \frac{105}{2}\Delta_1 h_3^2 E_2 + 3\Delta_4 E_0 E_2^2 + 6\Delta_2 E_2 h_4 + 630 h_4^2 E_1 h_1\\
&+14805 h_4 h_1 E_3 h_2 + 4977 h_4 h_3 E_2 h_1 + 735 h_4 h_3 E_1 h_2 + 4032 h_4^2 h_0 E_2 + 19656 h_4 h_3 h_0 E_3\\
&+120\Delta_1 h_2 E_2 h_4 + 3\Delta_4 E_1^2 E_2 + 6\Delta_4 E_0 E_1 E_3 + \frac{10101}{2} h_3 h_2^2 E_3 + 315\Delta_1 h_1 E_3 h_4 + 30\Delta_1 h_3 h_4 E_1\\
&+\frac{525}{2}\Delta_1 h_2 E_3 h_3 + \frac{24759}{4} h_3^2 E_3 h_1 + \frac{3045}{2} h_3^2 E_2 h_2 = 0,\\
&1512 h_4 h_0^2 E_3 + 45 h_4 h_1^2 E_1 + 126 h_3 h_1^2 E_2 + 546 h_2^2 h_0 E_3 + 63 h_2^2 E_2 h_1 + 315 h_2 E_3 h_1^2\\
&+30\Delta_1 h_0 h_3 E_2 + 60\Delta_1 h_0 h_2 E_3 + \frac{9}{2}\Delta_1 h_1 E_1 h_3 + 15\Delta_1 h_1 E_2 h_2 + 12\Delta_1 h_0 h_4 E_1 - \Delta_3 E_1\\
&+45 h_3^2 E_1 h_0 + \Delta_1 h_2^2 E_1 + \Delta_2 E_1 h_2 + 3\Delta_2 E_2 h_1 + 6\Delta_2 h_0 E_3 + 3\Delta_4 E_0^2 E_1 + \frac{45}{2}\Delta_1 h_1^2 E_3 + E_1 h_2^3\\
&+132 h_4 E_1 h_0 h_2 + 756 h_4 h_0 E_2 h_1 + 420 h_3 h_2 h_0 E_2 + 27 h_3 h_2 E_1 h_1 + 1512 h_3 h_0 E_3 h_1 = 0,\\
&675 h_4 h_1^2 E_2 + 675 h_3^2 h_0 E_2 + 63 h_3^2 E_1 h_1 + \frac{63}{2} h_3 E_1 h_2^2 + \frac{2835}{2} h_3 E_3 h_1^2 + 135\Delta_1 h_0 E_3 h_3\\
&+42\Delta_1 h_1 h_3 E_2 + \frac{195}{2}\Delta_1 h_1 h_2 E_3 + \frac{15}{2}\Delta_1 h_2 E_1 h_3 + 72\Delta_1 h_0 E_2 h_4 + 15\Delta_1 h_1 h_4 E_1 + \frac{1995}{2} h_2^2 E_3 h_1\\
&+16\Delta_1 h_2^2 E_2 + \frac{3}{2}\Delta_2 E_1 h_3 + 4\Delta_2 E_2 h_2 + \frac{15}{2}\Delta_2 E_3 h_1 + 678 h_3 h_2 E_2 h_1 + 3780 h_3 h_2 h_0 E_3 = 0,\\
&945 h_3^2 E_2 h_1 + 105 h_3^2 E_1 h_2 + 4455 h_3^2 h_0 E_3 + 665 h_3 h_2^2 E_2 + 90\Delta_1 h_1 E_2 h_4 + \frac{369}{2}\Delta_1 h_1 E_3 h_3\\
&+6\Delta_4 E_0 E_1 E_2 + 20\Delta_1 h_2 h_4 E_1 + 65\Delta_1 h_2 h_3 E_2 + 252\Delta_1 h_0 E_3 h_4 + 504 h_4^2 E_1 h_0 + \frac{15}{2}\Delta_1 h_3^2 E_1\\
&+2\Delta_2 h_4 E_1 + 5\Delta_2 h_3 E_2 + 9\Delta_2 h_2 E_3 + 3\Delta_4 E_0^2 E_3 + 81\Delta_1 h_2^2 E_3 + 729 E_3 h_2^3 + \Delta_4 E_1^3 - \Delta_3 E_3\\
&+3780 h_4 h_3 h_0 E_2 + 504 h_4 h_3 E_1 h_1 + 10548 h_4 h_2 h_0 E_3 + 2520 h_4 h_2 E_2 h_1\\
&+5823 h_3 h_1 E_3 h_2 + 182 h_4 E_1 h_2^2 + 4455 h_4 E_3 h_1^2 = 0.
\end{aligned}
\right\}\qquad (33)
$$

Set–1: If we set $h_0 = h_1 = h_3 = 0$, in Eqs. (33), and solve it by using the Maple, we will get the following results:

Result–1:

$$E_0 = 0, E_1 = 0, E_2 = 0, E_3 = \pm 24 h_4 \sqrt{-\frac{35 h_4}{\Delta_4}}, h_2 = -\frac{\Delta_1}{83}, \tag{34}$$

and

$$\Delta_2 = \frac{1891 \Delta_1^2}{6889}, \Delta_3 = -\frac{11025 \Delta_1^3}{571787}, \tag{35}$$

providing $h_4 \Delta_4 < 0$. By substituting (34) along with (23) and (24) into (32), Eqs. (1) and (2) have the solutions:

(I): Bright soliton:

$$u(x,t) = \pm \frac{24\Delta_1}{6889} \sqrt{-\frac{2905 \Delta_1}{\Delta_4}} \times sech^3\left(\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{36}$$

$$v(x,t) = \pm \frac{24 A \Delta_1}{6889} \sqrt{-\frac{2905 \Delta_1}{\Delta_4}} sech^3\left(\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{37}$$

provided $\Delta_1 < 0$ and $\Delta_4 > 0$.

(II) Singular soliton:

$$u(x,t) = \pm \frac{24\Delta_1}{6889} \sqrt{\frac{2905 \Delta_1}{\Delta_4}} \times csch^3\left(\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{38}$$

$$v(x,t) = \pm \frac{24 A \Delta_1}{6889} \sqrt{\frac{2905 \Delta_1}{\Delta_4}} \times csch^3\left(\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{39}$$

provided $\Delta_1 < 0$ and $\Delta_4 < 0$.

Result–2:

$$E_0 = 0, E_1 = \pm \frac{288 \Delta_1}{581} \sqrt{-\frac{35 h_4}{\Delta_4}}, E_2 = 0, \quad E_3 = \pm 24 \sqrt{-\frac{35 h_4}{\Delta_4}} h_4, \quad h_2 = \frac{17 \Delta_1}{581}, \tag{40}$$

and

$$\Delta_2 = \frac{1891 \Delta_1^2}{6889}, \Delta_3 = \frac{1748025 \Delta_1^3}{196122941}, \tag{41}$$

provided $h_4 \Delta_4 < 0$. By substituting (40) along with (23) and (24) into (32), Eqs. (1) and (2) have the solutions:

(I): Bright solitons:

$$u(x,t) = \pm \frac{24\Delta_1}{48223} \sqrt{\frac{7055 \Delta_1}{\Delta_4}} sech\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right) \times \left[12 - 17\, sech^2\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{42}$$

$$v(x,t) = \pm \frac{24 A \Delta_1}{48223} \sqrt{\frac{7055 \Delta_1}{\Delta_4}} sech\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right) \times \left[12 - 17\, sech^2\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{43}$$

provided $\Delta_1 > 0$ and $\Delta_4 > 0$.

(II): Singular solitons:

$$u(x,t) = \pm \frac{24\Delta_1}{48223} \sqrt{-\frac{7055 \Delta_1}{\Delta_4}} csch\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right) \times \left[12 + 17\, csch^2\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{44}$$

$$v(x,t) = \pm \frac{24 A \Delta_1}{48223} \sqrt{-\frac{7055 \Delta_1}{\Delta_4}} csch\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right) \times \left[12 + 17\, csch^2\left(\frac{\sqrt{9877 \Delta_1}}{581}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{45}$$

provided $\Delta_1 > 0$ and $\Delta_4 < 0$.

Set–2: If we set $h_1 = h_3 = 0$ and $h_0 = \frac{h_2^2}{4h_4}$, in Eqs. (33), and solving it by using the Maple we will get the following result:

$$E_0 = 0, E_1 = \pm \frac{18 \Delta_1}{83} \sqrt{-\frac{35 h_4}{\Delta_4}}, E_2 = 0, \quad E_3 = \pm 24 h_4 \sqrt{-\frac{35 h_4}{\Delta_4}}, h_2 = \frac{\Delta_1}{166}, \tag{46}$$

and

$$\Delta_2 = \frac{946 \Delta_1^2}{6889}, \Delta_3 = \frac{1260 \Delta_1^3}{571787}, \tag{47}$$

provided $h_4 \Delta_4 < 0$. By substituting (46) along with (25) and (26) into (32), Eqs. (1) and (2) have the solutions:

(I): Dark solitons:

$$u(x,t) = \pm \frac{3\Delta_1}{83} \sqrt{\frac{35 \Delta_1}{83 \Delta_4}} \tanh\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \times \left[3 - \tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{48}$$

$$v(x,t) = \pm \frac{3 \Im \Delta_1}{83} \sqrt{\frac{35 \Delta_1}{83 \Delta_4}} \tanh\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right) \times \left[3 - \tanh^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x - ct)\right)\right] \times \exp[i(-\kappa x + \Omega t + \varsigma_0)], \tag{49}$$

provided $\Delta_1 < 0$ and $\Delta_4 < 0$.

(II) Singular solitons:

$$u(x,t) = \pm\frac{3\Delta_1}{83}\sqrt{\frac{35\Delta_1}{83\Delta_4}}\coth\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x-ct)\right)\times \left[3-\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x-ct)\right)\right]\exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{50}$$

$$v(x,t) = \pm\frac{3A\Delta_1}{83}\sqrt{\frac{35\Delta_1}{83\Delta_4}}\coth\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x-ct)\right)\times \left[3-\coth^2\left(\frac{1}{2}\sqrt{-\frac{\Delta_1}{83}}(x-ct)\right)\right]\exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{51}$$

provided $\Delta_1 < 0$ and $\Delta_4 < 0$.

Set–3: If we set $h_1 = h_3 = 0$, in Eqs. (33), and solving it by using the Maple, we will get the following result:

$$E_0 = 0, E_1 = 0, E_2 = 0, E_3 = \pm 24h_4\sqrt{-\frac{35h_4}{\Delta_4}}, \quad h_0 = \frac{32}{241115}\frac{\Delta_1^2}{h_4}, h_2 = -\frac{\Delta_1}{83}, \tag{52}$$

and

$$\Delta_2 = \frac{2543\Delta_1^2}{34445}, \Delta_3 = \frac{381357\Delta_1^3}{20012545}, \tag{53}$$

provided $h_4\Delta_4 < 0$. By substituting (52) along with (30) and (31) into (32), Eqs. (1) and (2) have the Weierstrass elliptic function solutions:

(I):

$$u(x,t) = \pm 648\sqrt{-\frac{35}{\Delta_4}}\left[\frac{83\wp\left((x-ct),\frac{419\Delta_1^2}{2893380},-\frac{1117\Delta_1^3}{4322709720}\right)}{498\wp\left((x-ct),\frac{419\Delta_1^2}{2893380},-\frac{1117\Delta_1^3}{4322709720}\right)-\Delta_1}\right]^3 \times \exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{54}$$

$$v(x,t) = \pm 648A\sqrt{-\frac{35}{\Delta_4}}\left[\frac{83\wp\left((x-ct),\frac{419\Delta_1^2}{2893380},-\frac{1117\Delta_1^3}{4322709720}\right)}{498\wp\left((x-ct),\frac{419\Delta_1^2}{2893380},-\frac{1117\Delta_1^3}{4322709720}\right)-\Delta_1}\right]^3 \times \exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{55}$$

provided $\Delta_4 < 0$.

(II):

$$u(x,t) = \pm\frac{8}{3}\sqrt{-\frac{105}{\Delta_4}}\times \left[3\wp\left((x-ct),-\frac{244\Delta_1^2}{723345},-\frac{872\Delta_1^3}{540338715}\right)+\frac{1}{83}\Delta_1\right]^{\frac{3}{2}} \times \exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{56}$$

$$v(x,t) = \pm\frac{8}{3}A\sqrt{-\frac{105}{\Delta_4}}\times \left[3\wp\left((x-ct),-\frac{244\Delta_1^2}{723345},-\frac{872\Delta_1^3}{540338715}\right)+\frac{1}{83}\Delta_1\right]^{\frac{3}{2}} \times \exp[i(-\kappa x+\Omega t+\varsigma_0)], \tag{57}$$

provided $\Delta_4 < 0$.

4. Conclusions

The work in this paper addressed to highly dispersive optical solitons with differential group delay with an account of the Kerr law of self-phase modulation using the extended auxiliary equation approach. This allows to describe the full spectrum of optical 1-soliton solutions within the model that are all exhibited. The corresponding parameter constraints are also enlisted for these respective solitons to exist. The work is thus promising and can be continued. In the future, this model will be extended to dispersion-flattened fibers and those results will be recovered and disseminated after comparing with the pre-existing works. Additionally, the perturbation terms will be included so that a complete picture of the model will be painted. Moreover, highly dispersive optical solitons would be addressed with other optoelectronic devices apart from optical fibers. This is just the tip of the iceberg.

Disclosure

The authors claim that there is no conflict of interest.

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Високодисперсійні оптичні солітони з диференціальною груповою затримкою для закону Керра фазової самомодуляції

R.M.A. Shohib, M.E.M. Alngar, Y. Yildirim & A. Biswas

Анотація. У статті розглядаються високодисперсійні оптичні солітони з диференціальною груповою затримкою, що мають закон Керра фазової самомодуляції. Підхід розширеного допоміжного рівняння відновлює повний спектр солітонів. Також представлено розв’язання у вигляді еліптичних функцій Веєрштрасса. Усі результати представлені з необхідними обмеженнями параметрів, які природно випливають із схеми.

Ключові слова: солітони, дисперсія, подвійне променезаломлення.