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# Implicit quiescent optical soliton perturbation with nonlinear chromatic dispersion and generalized temporal evolution having a plethora of self-phase modulation structures by Lie symmetry

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**Abstract.** This paper recovers implicit quiescent perturbed optical solitons that emerge from the nonlinear Schrödinger equation with Hamiltonian perturbation terms having arbitrary intensity. The model is considered in the context of generalized temporal evolution and nonlinear chromatic dispersion. Eighteen forms of self-phase modulation structures are taken into account. The integration is carried out by the implementation of Lie symmetry. The parameter constraints that guarantee the existence of these solitons are also presented.

**Keywords**: nonlinear dispersion, stationary solitons.

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## 1. Introduction

The dynamics of quiescent optical solitons is one of the non-essential features in optoelectronics. This unwanted feature is detrimental to the soliton transmission technology and is therefore a telecommunication engineer's nightmare. This fallout happens when the delicate balance between dispersion and nonlinearity is lost during soliton transmission across intercontinental distances under the ocean or underground. Therefore, the balance must be maintained throughout all communication. One of the sources of this fallout is that chromatic dispersion (CD) is rendered to be nonlinear as opposed to being linear when the solitons stay mobile. The rough handling of fibers, as well as fiber bends and twists, lead to the pulses propagating through the fiber getting stalled, which leads to the formation of these quiescent solitons.

Various mathematical approaches have been implemented in the past to recover quiescent optical solitons from a wide range of models in optical fibers, magneto-optic waveguides, optical couplers and others [1–15]. However, this paper addresses the formation of these quiescent optical solitons for the governing nonlinear Schrödinger's equation (NLSE) that is considered with a few Hamiltonian perturbation terms, which appear with arbitrary intensity. The model is addressed with eighteen forms of self-phase modulation (SPM) structure that appear in quantum optics in the context of optical fibers. The temporal evolution is also taken to be in its generalized form; thus, the case of linear temporal evolution collapses to its special case that was already addressed earlier [1]. The unperturbed version of the model was addressed earlier for five forms of SPM structure, both with linear temporal evolution as well as generalized temporal evolution. This paper is thus an extension/generalization of the previously reported works. The integration algorithm, which is adopted in the paper, is Lie symmetry as in previous works. The details of the analysis and the results are presented and derived after a succinct introduction to the model.

### 1.1. Governing model

The dimensionless form of the NLSE with the generalized temporal evolution and nonlinear CD and the non–Kerr law of SPM is structured as:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + F(|q|^2)q^l = \\ = i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m}(q^l)_x]. \quad (1)$$

In equation (1), the dependent variable $q(x, t)$ stands for the wave amplitude, while the independent variables $x$ and $t$ represent the spatial and temporal coordinates, respectively. The first term represents temporal evolution where $l$ is the generalized temporal evolution parameter and $i = \sqrt{-1}$. The second term is the nonlinear CD, where $n$ represents the parameter of nonlinearity for CD. If $n = 0$ together with $l = 1$, CD collapses to its linear version, in which case, the solitons would be mobile. In the third term, $F$ represents the generalized functional form of the non-Ker law of SPM. On the right-hand side, the perturbation terms represent self-steepening effect and self-frequency shift whose coefficients are given by $\lambda$ and $\theta_j$ for $j = 1, 2$, respectively. Here, the parameter $m$ represents arbitrary intensity. For $m = 0$, one recovers the actual intensity of light.

### 2. Mathematical analysis

To analyze equation (1) the following structure of the quiescent solitons is selected:

$$q(x,t) = \phi(x)\, e^{i\omega t}, \quad (2)$$

where $\omega$ is the wave number and $\phi$ represents the amplitude form of the wave. Substituting into (1) and decomposing into real and imaginary parts lead to the equations

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \\ \times\{\phi'(x)\}^2 + \phi^2(x)[F\{\phi^2(x)\} - l\omega] = 0, \quad (3)$$

and

$$\lambda(2m+l) + 2m\theta_1 + l\theta_2 = 0, \quad (4)$$

respectively.

Equation (4) serves as the parameter constraints between the perturbation terms, the generalized temporal evolution parameter and the arbitrary intensity parameter. Eq. (3) is the ordinary differential equation (ODE) that is going to be addressed for eighteen different structures of SPM, denoted by the functional $F$, and the results for the quiescent solitons will be derived and exhibited in the subsequent section. These results will be with additional parameter constraints that will naturally emerge for various forms of SPM structures.

### 3. Application to several SPM structures

This section will carry out the integration of ODE from the previous section, given by (3), for eighteen different forms of SPM structures. The Lie symmetry analysis will be implemented for each of these models, and the established quiescent solitons will be presented.

### 3.1. Kerr law

For Kerr law of nonlinearity, the functional $F$ takes the form:

$$F(|q|^2) = b|q|^2, \quad (5)$$

where $b$ is a non-zero constant and the governing model therefore takes the form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + b|q|^2 q^l = \\ i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m}(q^l)_x], \quad (6)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \\ \times\{\phi'(x)\}^2 + \phi^2(x)\{b\phi^2(x) - l\omega\} = 0. \quad (7)$$

The ODE (7), along with the constraint condition (4), permits a translational Lie symmetry $\partial/\partial x$ that integrates (7) to

$$x = \pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(l+n)(2l+n)}{l\omega}}\ {}_2F_1\left(\frac{1}{2}, \frac{n}{4}; \frac{4+n}{4}; \frac{b(2l+n)\phi^2}{l(2+2l+n)\omega}\right), \quad (8)$$

where the Gauss hypergeometric function is defined as

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \quad (9)$$

with the Pochhammer symbol being

$$(p)_n = \begin{cases} 1 & n = 0, \\ p(p+1)\cdots(p+n-1) & n > 0. \end{cases} \quad (10)$$

The condition that guarantees convergence of the hypergeometric series is

$$|z| < 1, \quad (11)$$

which for (9) implies

$$-\left|\frac{\omega(2l+n+2)}{b(2l+n)}\right|^{\frac{1}{2}} < \phi(x) < \left|\frac{\omega(2l+n+2)}{b(2l+n)}\right|^{\frac{1}{2}}. \quad (12)$$

Finally (9) also compels the parameter constraint:

$$a\omega > 0. \quad (13)$$

### 3.2. Power law

For power-law of nonlinearity the functional $F$ is taken to be

$$F(|q|^2) = b|q|^{2m}, \quad (14)$$

for non-zero constant $b$ and therefore the governing model takes the form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + b|q|^{2m} q^l = \\ i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m}(q^l)_x], \quad (15)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \\ \times\{\phi'(x)\}^2 + \phi^2(x)\{b\phi^{2m}(x) - l\omega\} = 0. \quad (16)$$

The ODE (16) along with the constraint condition (4) permits a translational Lie symmetry $\partial/\partial x$ that integrates (16) to reveal the implicit quiescent optical soliton as

$$x = \pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(l+n)(2l+n)}{l\omega}}\; {}_2F_1\left(\frac{1}{2},\frac{n}{4m};1+\frac{n}{4m};\frac{b(2l+n)\phi^{2m}}{l(2(l+m)+n)\omega}\right). \quad (17)$$

The condition given by (11) for the Gauss hypergeometric function to converge translates to

$$-\left|\frac{\omega(2(l+m)+n)}{b(2l+n)}\right|^{\frac{1}{2m}} < \phi(x) < \left|\frac{\omega(2(l+m)+n)}{b(2l+n)}\right|^{\frac{1}{2m}}, \quad (18)$$

while the constraint (13) still holds.

### 3.3. Parabolic (cubic-quintic) law

For parabolic law of nonlinearity, the functional $F$ takes the form:

$$F(|q|^2) = b_1|q|^2 + b_2|q|^4, \quad (19)$$

where $b_j$ for $j = 1, 2$ are nonzero constants. Therefore, the governing model takes the form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^2 + b_2|q|^4)q^l = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (20)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x)\times\{\phi'(x)\}^2 + \phi^2(x)\{b_1\phi^2(x) + b_2\phi^4(x) - l\omega\} = 0. \quad (21)$$

The ODE (7), along with the constraint condition (4), permits a translational Lie symmetry $\partial/\partial x$ that integrates (7) to

$$x = \pm\frac{\phi^{\frac{n}{2}}}{n}\sqrt{\frac{2a(l+n)(2l+n)}{l\omega}}\times F_1\left(\frac{n}{4};\frac{1}{2},\frac{1}{2};\frac{n+4}{4};-\frac{2(2l+n)(2l+n+2)\phi^2 b_2}{b_1B_1+B_2},-\frac{2(2l+n)(2l+n+2)\phi^2 b_2}{b_1B_1-B_2}\right), \quad (22)$$

where

$$B_1 = (2l+n)(2l+n+4), \quad (23)$$

and

$$B_2 = \sqrt{B_1\{b_1^2 B_1 + 4b_2 l\omega(2l+n+2)^2\}}. \quad (24)$$

Here, the Appell hypergeometric function of two variables is defined by the infinite series:

$$F_1(a;b_1,b_2;c;x,y) = = x^m y^n\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!\,n!}\right), \quad (25)$$

which is convergent inside the region

$$\max(|x|,|y|) < 1, \quad (26)$$

which in this case, (22) implies

$$\max\left(\left|\frac{(2l+n)(2l+n+2)\phi^2 b_2}{b_1B_1+B_2}\right|,\left|\frac{(2l+n)(2l+n+2)\phi^2 b_2}{b_1B_1-B_2}\right|\right) < 1. \quad (27)$$

One would also require from (27)

$$b_1B_1 \pm B_2 \neq 0, \quad (28)$$

for the implicit quiescent solitons to exist.

### 3.4. Dual-power law

For dual-power law, the SPM is structured as

$$F(|q|^2) = b_1|q|^{2m} + b_2|q|^{2m+2}, \quad (29)$$

for constants $b_j$ with $j = 1, 2$. In this case the perturbed NLSE is

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^{2m} + b_2|q|^{2m+2})q^l = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (30)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x)\times\{\phi'(x)\}^2 + \phi^2(x)\{b_1\phi^{2m}(x) + b_2\phi^{2m+2}(x) - l\omega\} = 0. \quad (31)$$

The translational Lie symmetry leads to the integral of the ODE to give the implicit quiescent optical solitons in quadratures as

$$x = \pm\int\sqrt{-\frac{A}{B}}d\phi, \quad (32)$$

where

$$A = a(l+n)(2l+n)\{2(l+m)+n\}\times\{2(l+m+1)+n\}\phi^{n-2}, \quad (33)$$

and

$$B = 2\begin{bmatrix}-l\{2(l+m)+n\}\{2(1+l+m)+n\}\omega + \\ (2l+n)\phi^{2m}\times \\ [\{2(l+m+1)+n\}b_1 + \{2(l+m)+n\}\phi^2 b_2]\end{bmatrix}. \quad (34)$$

The criteria for the quiescent solitons to exist is given by

$$AB < 0. \quad (35)$$

### 3.5. Polynomial (cubic-quintic-septic) law

For polynomial law of SPM, one writes the SPM structure as

$$F(|q|^2) = b_1|q|^2 + b_2|q|^4 + b_3|q|^6, \quad (36)$$

where $b_j$ for $j = 1, 2, 3$ are nonzero constants. Therefore, the NLSE with the perturbation terms shape up as

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^2 + b_2|q|^4 + b_3|q|^6)q^l = = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (37)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n) \times \phi^{n}(x)\{\phi'(x)\}^2 + \phi^2(x) \times \{b_1\phi^2(x) + b_2\phi^4(x) + b_3\phi^6(x) - l\omega\} = 0. \quad (38)$$

The ODE (38), along with the constraint condition (4), permits a translational Lie symmetry $\partial/\partial x$ that integrates it to give the implicit quiescent optical solitons, which is in terms of quadratures as given by (32), where in this case:

$$A = 16al^5\phi^{n-2} + 96al^4\phi^{n-2} + 48al^4n\phi^{n-2} + 56al^3n^2\phi^{n-2} + 176al^3\phi^{n-2} + 240al^3n\phi^{n-2} + 32al^2n^3\phi^{n-2} + 216al^2n^2\phi^{n-2} + 96al^2\phi^{n-2} + 352al^2n\phi^{n-2} + 9aln^4\phi^{n-2} + 84aln^3\phi^{n-2} + 220aln^2\phi^{n-2} + 144aln\phi^{n-2} + an^5\phi^{n-2} + 12an^4\phi^{n-2} + 44an^3\phi^{n-2} + 48an^2\phi^{n-2}, \quad (39)$$

$$B = 16b_3l^3\phi^6 + 16b_2l^3\phi^4 + 16b_1l^3\phi^2 + 24b_3l^2n\phi^6 + 24b_2l^2n\phi^4 + 24b_1l^2n\phi^2 + 48b_3l^2\phi^6 + 64b_2l^2\phi^4 + 80b_1l^2\phi^2 + 12b_3ln^2\phi^6 + 12b_2ln^2\phi^4 + 12b_1ln^2\phi^2 + 48b_3ln\phi^6 + 64b_2ln\phi^4 + 80b_1ln\phi^2 + 32b_3l\phi^6 + 48b_2l\phi^4 + 96b_1l\phi^2 + 2b_3n^3\phi^6 + 2b_2n^3\phi^4 + 2b_1n^3 + 12b_3n^2\phi^6 + 16b_2n^2\phi^4 + 20b_1n^2\phi^2 + 16b_3n\phi^6 + 24b_2n\phi^4 + 48b_1n\phi^2 - 16l^4\omega - 24l^3n\omega - 96l^3\omega - 12l^2n^2\omega - 96l^2n\omega - 176l^2\omega - 2ln^3\omega - 24ln^2\omega - 88ln\omega - 96l\omega. \quad (40)$$

The same constraint condition for the quiescent solitons to exist as given by (35) is also valid in this case.

**3.6. Triple-power law**

For triple power law, the structure of SPM takes the form:

$$F(|q|^2) = b_1|q|^{2m} + b_2|q|^{2m+2} + b_3|q|^{2m+4}, \quad (41)$$

with constants $b_j$ for $j = 1, 2, 3$. This gives the perturbed NLSE as

$$i(q^l)_t + a(|q|^n q^l)_{xx} + +(b_1|q|^{2m} + b_2|q|^{2m+2} + b_3|q|^{2m+4})q^l = = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (42)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \times\{\phi'(x)\}^2 + \phi^2(x) \times \{b_1\phi^{2m}(x) + b_2\phi^{2m+2}(x) + b_3\phi^{2m+4}(x) - l\omega\} = 0. \quad (43)$$

The translational Lie symmetry leads to the integral of the ODE to give the implicit quiescent optical solitons in quadratures as given by (32) with the constraint as in (35) where for this situation

$$A = 16al^5\phi^{n-2} + 48al^4m\phi^{n-2} + 48al^4\phi^{n-2} + 48al^4n\phi^{n-2} + 48al^3m^2\phi^{n-2} + 96al^3m\phi^{n-2} + 120al^3mn\phi^{n-2} + 56al^3n^2\phi^{n-2} + 32al^3\phi^{n-2} + 120al^3n\phi^{n-2} + 16al^2m^3\phi^{n-2} + 48al^2m^2\phi^{n-2} + 96al^2m^2n\phi^{n-2} + 108al^2mn^2\phi^{n-2}32al^2m\phi^{n-2} + 192al^2mn\phi^{n-2} + 32al^2n^3\phi^{n-2} + 108al^2n^2\phi^{n-2} + 64al^2n\phi^{n-2} + 24alm^3n\phi^{n-2}60alm^2n^2\phi^{n-2} + 72alm^2n\phi^{n-2} + 42almn^3\phi^{n-2} + 120almn^2\phi^{n-2} + 48almn\phi^{n-2} + 9aln^4\phi^{n-2} + 42aln^3\phi^{n-2} + 40aln^2\phi^{n-2}8am^3n^2\phi^{n-2} + 12am^2n^3\phi^{n-2} + 24am^2n^2\phi^{n-2} + 6amn^4\phi^{n-2} + 24amn^3\phi^{n-2}16amn^2\phi^{n-2}an^5\phi^{n-2} + 6an^4\phi^{n-2} + 8an^3\phi^{n-2}, \quad (44)$$

$$B = 16b_1l^3\phi^{2m} + 16b_2l^3\phi^{2m+2} + 16b_3l^3\phi^{2m+4} + 24b_1l^2n\phi^{2m} + 24b_2l^2n\phi^{2m+2} + 24b_3l^2n\phi^{2m+4} + 48b_1l^2\phi^{2m} + 32b_1l^2m\phi^{2m} + 32b_2l^2\phi^{2m+2} + 32b_2l^2m\phi^{2m+2} + 16b_3l^2\phi^{2m+4} + 32b_3l^2m\phi^{2m+4} + 16b_1lm^2\phi^{2m} + 16b_2lm^2\phi^{2m+2} + 16b_3lm^2\phi^{2m+4} + 12b_1ln^2\phi^{2m} + 12b_2ln^2\phi^{2m+2} + 12b_3ln^2\phi^{2m+4} + 48b_1ln\phi^{2m} + 32b_1lmn\phi^{2m} + 32b_2ln\phi^{2m+2} + 32b_2lmn\phi^{2m+2} + 16b_3ln\phi^{2m+4} + 32b_3lmn\phi^{2m+4} + 32b_1l\phi^{2m} + 48b_1lm\phi^{2m} + 32b_2lm\phi^{2m+2} + 16b_3lm\phi^{2m+4} + 8b_1m^2n\phi^{2m} + 8b_2m^2n\phi^{2m+2} + 8b_3m^2n\phi^{2m+4} + 2b_1n^3\phi^{2m} + 2b_2n^3\phi^{2m+2} + 2b_3n^3\phi^{2m+4} + 8b_1mn^2\phi^{2m} + 12b_1n^2\phi^{2m} + 8b_2mn^2\phi^{2m+2} + 8b_2n^2\phi^{2m+2} + 8b_3mn^2\phi^{2m+4} + 4b_3n^2\phi^{2m+4} + 24b_1mn\phi^{2m} + 16b_1n\phi^{2m} + 16b_2mn\phi^{2m+2} + 8b_3mn\phi^{2m+4} - 16l^4\omega - 48l^3m\omega - 24l^3n\omega - 48l^3\omega - 48l^2m^2\omega - 48l^2mn\omega - 96l^2m\omega - 12l^2n^2\omega - 48l^2n\omega - 32l^2\omega - 16lm^3\omega - 24lm^2n\omega - 48lm^2\omega - 12lmn^2\omega - 48lmn\omega - 32lm\omega - 2ln^3\omega - 12ln^2\omega - 16ln\omega. \quad (45)$$

**3.7. Cubic-quintic-septic-nonic law**

For cubic-quintic-septic-nonic law of SPM, the nonlinear refractive index change is given as

$$F(|q|^2) = b_1|q|^2 + b_2|q|^4 + b_3|q|^6 + b_4|q|^8, \quad (46)$$

where $b_j$ for $1 \le j \le 4$ are real-valued constants. With this form of SPM the governing NLSE is

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^2 + b_2|q|^4 + b_3|q|^6 + b_4|q|^8)q^l = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (47)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \{\phi'(x)\}^2 + \phi^2(x)\{b_1\phi^2(x) + b_2\phi^4(x) + b_3\phi^6(x) + b_4\phi^8(x) - l\omega\} = 0. \quad (48)$$

This ODE (48), along with the constraint condition (4), permits the translational Lie symmetry as in Kerr law. When implemented into the above ODE, the implicit quiescent optical solitons emerge as

$$x = \pm \int \frac{1}{\phi} \sqrt{\frac{a(l+n)\phi^n}{2\left(\frac{l\omega}{2l+n} - \frac{b_1\phi^2}{2l+n+2} - \frac{b_2\phi^4}{2l+n+4} - \frac{b_3\phi^6}{2l+n+6} - \frac{b_4\phi^8}{2l+n+8}\right)}}\, d\phi. \tag{49}$$

### 3.8. Quadrupled-power law

For quadrupled power-law of nonlinear refractive index change, the SPM is structured as

$$F(|q|^2) = b_1|q|^{2m} + b_2|q|^{2m+2} + b_3|q|^{2m+4} + b_4|q|^{2m+6}, \tag{50}$$

for non-zero real-valued constants $b_j$ with $1 \le j \le 4$. For this form of SPM, the governing NLSE takes the form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^{2m} + b_2|q|^{2m+2} + b_3|q|^{2m+4} + b_4|q|^{2m+6})q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \tag{51}$$

while the ODE (3) simplifies to:

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n) \times \phi^n(x)\{\phi'(x)\}^2 \phi^2(x)\{b_1\phi^{2m}(x) + b_2\phi^{2m+2}(x) + b_3\phi^{2m+4}(x) + b_4\phi^{2m+6}(x) - l\omega\} = 0. \tag{52}$$

The translational Lie symmetry leads to the integral of the ODE to give the implicit quiescent optical solitons in quadratures as given by (32) with the constraint as in (35) with $A$ and $B$ given by

$$A = 32al^6\phi^{n-2} + an^6\phi^{n-2} + 192al^5\phi^{n-2} + 12an^5\phi^{n-2} + 11aln^5\phi^{n-2} + 8amn^5\phi^{n-2} + 352al^4\phi^{n-2} + 32al^2m^4\phi^{n-2} + 50al^2n^4\phi^{n-2} + 24am^2n^4\phi^{n-2} + 44an^4\phi^{n-2} + 108aln^4\phi^{n-2} + 72amn^4\phi^{n-2} + 72almn^4\phi^{n-2} + 192al^3\phi^{n-2} + 128al^3m^3\phi^{n-2} + 192al^2m^3\phi^{n-2} + 120al^3n^3\phi^{n-2} + 32am^3n^3\phi^{n-2} + 384al^2n^3\phi^{n-2} + 144am^2n^3\phi^{n-2} + 168alm^2n^3\phi^{n-2} + 48an^3\phi^{n-2} + 308aln^3\phi^{n-2} + 256al^2mn^3\phi^{n-2} + 176amn^3\phi^{n-2} + 504almn^3\phi^{n-2} + 192al^4m^2\phi^{n-2} + 576al^3m^2\phi^{n-2} + 352al^2m^2\phi^{n-2} + 160al^4n^2\phi^{n-2} + 16am^4n^2\phi^{n-2} + 672al^3n^2\phi^{n-2} + 96am^3n^2\phi^{n-2} + 160alm^3n^2\phi^{n-2} + 792al^2n^2\phi^{n-2} + 432al^2m^2n^2\phi^{n-2} + 176am^2n^2\phi^{n-2} + 720alm^2n^2\phi^{n-2} + 240aln^2\phi^{n-2} + 448al^3mn^2\phi^{n-2} + 1296al^2mn^2\phi^{n-2} + 96amn^2\phi^{n-2} + 880almn^2\phi^{n-2} + 128al^5m\phi^{n-2}576al^4m\phi^{n-2} + 704al^3m\phi^{n-2} + 192al^2m\phi^{n-2} + 112al^5n\phi^{n-2} + 576al^4n\phi^{n-2} + 48alm^4n\phi^{n-2} + 880al^3n\phi^{n-2} + 256al^2m^3n\phi^{n-2} + 288alm^3n\phi^{n-2} + 384al^2n\phi^{n-2} + 480al^3m^2n\phi^{n-2} + 1152al^2m^2n\phi^{n-2} + 528alm^2n\phi^{n-2} + 384al^4mn\phi^{n-2} + 1440al^3mn\phi^{n-2} + 1408al^2mn\phi^{n-2} + 288almn\phi^{n-2}, \tag{53}$$

$$B = 32l^4b_1\phi^{2m} + 2n^4b_1\phi^{2m} + 192l^3b_1\phi^{2m} + 32lm^3b_1\phi^{2m} + 16ln^3b_1\phi^{2m} + 12mn^3b_1\phi^{2m} + 24n^3b_1\phi^{2m} + 352l^2b_1\phi^{2m} + 96l^2m^2b_1\phi^{2m} + 192lm^2b_1\phi^{2m} + 48l^2n^2b_1\phi^{2m} + 24m^2n^2b_1\phi^{2m} + 144ln^2b_1\phi^{2m} + 72lmn^2b_1\phi^{2m} + 96mn^2b_1\phi^{2m} + 88n^2b_1\phi^{2m} + 192lb_1\phi^{2m} + 96l^3mb_1\phi^{2m} + 384l^2mb_1\phi^{2m} + 352lmb_1\phi^{2m} + 64l^3nb_1\phi^{2m} + 16m^3nb_1\phi^{2m} + 288l^2nb_1\phi^{2m} + 96lm^2nb_1\phi^{2m} + 96m^2nb_1\phi^{2m} + 352lnb_1\phi^{2m} + 144l^2mnb_1\phi^{2m} 384lmnb_1\phi^{2m} + 176mnb_1\phi^{2m} + 96nb_1\phi^{2m} + 32l^4b_2\phi^{2m+2} + 2n^4b_2\phi^{2m+2} + 160l^3b_2\phi^{2m+2} + 32lm^3b_2\phi^{2m+2} + 16ln^3b_2\phi^{2m+2} + 12mn^3b_2\phi^{2m+2} + 20n^3b_2\phi^{2m+2} + 192l^2b_2\phi^{2m+2} + 96l^2m^2b_2\phi^{2m+2} + 160lm^2b_2\phi^{2m+2} + 48l^2n^2b_2\phi^{2m+2} + 24m^2n^2b_2\phi^{2m+2} + 120ln^2b_2\phi^{2m+2} + 72lmn^2b_2\phi^{2m+2} + 80mn^2b_2 + 48n^2b_2\phi^{2m+2} + 96l^3mb_2\phi^{2m+2} + 320l^2mb_2\phi^{2m+2} + 192lmb_2\phi^{2m+2} + 64l^3nb_2\phi^{2m+2} + 16m^3nb_2\phi^{2m+2} + 240l^2nb_2\phi^{2m+2} + 96lm^2nb_2\phi^{2m+2} + 80m^2nb_2\phi^{2m+2} + 192lnb_2\phi^{2m+2} + 144l^2mnb_2\phi^{2m+2} + 320lmnb_2\phi^{2m+2} + 96mnb_2\phi^{2m+2} + 32l^4b_3\phi^{2m+4} + 2n^4b_3\phi^{2m+4} + 128l^3b_3\phi^{2m+4} + 32lm^3b_3\phi^{2m+4} + 16ln^3b_3\phi^{2m+4} + 12mn^3b_3\phi^{2m+4} + 16n^3b_3\phi^{2m+4} + 96l^2b_3\phi^{2m+4} + 96l^2m^2b_3\phi^{2m+4} + 128lm^2b_3\phi^{2m+4} + 48l^2n^2b_3\phi^{2m+4} + 24m^2n^2b_3\phi^{2m+4} + 96ln^2b_3\phi^{2m+4} + 72lmn^2b_3\phi^{2m+4} + 64mn^2b_3\phi^{2m+4} + 24n^2b_3\phi^{2m+4} + 96l^3mb_3\phi^{2m+4} + 256l^2mb_3\phi^{2m+4} + 96lmb_3\phi^{2m+4} + 64l^3nb_3\phi^{2m+4} + 16m^3nb_3\phi^{2m+4} + 192l^2nb_3\phi^{2m+4} + 96lm^2nb_3\phi^{2m+4} + 64m^2nb_3\phi^{2m+4} + 96lnb_3\phi^{2m+4} + 144l^2mnb_3\phi^{2m+4} + 256lmnb_3\phi^{2m+4} + 48mnb_3\phi^{2m+4} + 32l^4b_4\phi^{2m+6} + 2n^4b_4\phi^{2m+6} + 96l^3b_4\phi^{2m+6} + 32lm^3b_4\phi^{2m+6} + 16ln^3b_4\phi^{2m+6} + 12mn^3b_4\phi^{2m+6} + 12n^3b_4\phi^{2m+6} + 64l^2b_4\phi^{2m+6} + 96l^2m^2b_4\phi^{2m+6} + 96lm^2b_4\phi^{2m+6} + 48l^2n^2b_4\phi^{2m+6} + 24m^2n^2b_4\phi^{2m+6} + 72ln^2b_4\phi^{2m+6} + 72lmn^2b_4\phi^{2m+6} + 48mn^2b_4\phi^{2m+6} + 16n^2b_4\phi^{2m+6} + 96l^3mb_4\phi^{2m+6} + 192l^2mb_4\phi^{2m+6} + 64lmb_4\phi^{2m+6} + 64l^3nb_4\phi^{2m+6} + 16m^3nb_4\phi^{2m+6} + 144l^2nb_4\phi^{2m+6} + 96lm^2nb_4\phi^{2m+6} + 48m^2nb_4\phi^{2m+6} + 64lnb_4\phi^{2m+6} + 144l^2mnb_4\phi^{2m+6} 192lmnb_4\phi^{2m+6} + 32mnb_4\phi^{2m+6} - 32l^5\omega - 192l^4\omega - 32lm^4\omega - 2ln^4\omega - 352l^3\omega - 128l^2m^3\omega - 192lm^3\omega - 16l^2n^3\omega - 24ln^3\omega - 16lmn^3\omega - 192l^2\omega - 192l^3m^2\omega - 576l^2m^2\omega - 352lm^2\omega - 48l^3n^2\omega - 144l^2n^2\omega - 48lm^2n^2\omega - 88ln^2\omega - 96l^2mn^2\omega - 144lmn^2\omega - 128l^4m\omega - 576l^3m\omega - 704l^2m\omega - 192lm\omega - 4l^4n\omega - 288l^3n\omega - 64lm^3n\omega - 352l^2n\omega - 192l^2m^2n\omega - 6288lm^2n\omega - 96ln\omega - 192l^3mn\omega - 576l^2mn\omega - 352lmn\omega. \tag{54}$$

The parametric restriction given by (35) must hold for the quiescent solitons to exist.

### 3.9. Anti-cubic law

For anti-cubic law of SPM, the functional $F$ is given by

$$F(|q|^2) = \frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4, \tag{55}$$

for non-zero constants $b_j$ with $j$ =1, 2, 3. Therefore, the governing NLSE is written as

$$i(q^l)_t + a(|q|^n q^l)_{xx} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right) q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m} (q^l)_x], \tag{56}$$

so that the ODE given by (3) reads:

$$a(l+n)\phi^{n+2}(x)[(l+n-1)\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_3\phi^8(x) + b_2\phi^6(x) + b_1 - l\omega\phi^4(x) = 0. \tag{57}$$

By virtue of the translational Lie symmetry supported by the ODE (57), the implicit quiescent optical solitons is given by (32), along with the existence criteria (35), where, in this case

$$A = 16al^5\phi^{n+2} + 16al^4\phi^{n+2} + 48al^4n\phi^{n+2} + 56al^3n^2\phi^{n+2} - 64al^3\phi^{n+2} + 40al^3n\phi^{n+2} + 32al^2n^3\phi^{n+2} + 36al^2n^2\phi^{n+2} - 64al^2\phi^{n+2} - 128al^2n\phi^{n+2} + 9aln^4\phi^{n+2} + 14aln^3\phi^{n+2} - 80aln^2\phi^{n+2} - 96aln\phi^{n+2} + an^5\phi^{n+2} + 2an^4\phi^{n+2} - 16an^3\phi^{n+2} - 32an^2\phi^{n+2} \tag{58}$$

and

$$B = 16b_3l^3\phi^8 + 16b_2l^3\phi^6 + 16b_1l^3 + 24b_3l^2n\phi^8 + 24b_2l^2n\phi^6 + 24b_1l^2n - 16b_3l^2\phi^8 + 48b_1l^2 + 12b_3ln^2\phi^8 + 12b_2ln^2\phi^6 + 12b_1ln^2 - 16b_3ln\phi^8 + 48b_1ln - 32b_3l\phi^8 - 64b_2l\phi^6 + 32b_1l + 2b_3n^3\phi^8 + 2b_2n^3\phi^6 + 2b_1n^3 - 4b_3n^2\phi^8 + 12b_1n^2 - 16b_3n\phi^8 - 32b_2n\phi^6 + 16b_1n - 16l^4\omega\phi^4 - 24l^3n\omega\phi^4 - 16l^3\omega\phi^4 - 12l^2n^2\omega\phi^4 - 16l^2n\omega\phi^4 + 64l^2\omega\phi^4 - 2ln^3\omega\phi^4 - 4ln^2\omega\phi^4 + 32ln\omega\phi^4 + 64l\omega\phi^4. \tag{59}$$

### 3.10. Generalized anti-cubic law

For the generalized anti-cubic law of nonlinearity, the SPM structure reads:

$$F(|q|^2) = \frac{b_1}{|q|^{2(m+1)}} + b_2|q|^{2m} + b_3|q|^{2(m+1)}, \tag{60}$$

for non-zero constants $b_j$ with $1 \le j \le 3$. With this form of SPM, the governing NLSE looks:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + \left(\frac{b_1}{|q|^{2(m+1)}} + b_2|q|^{2m} + b_3|q|^{2(m+1)}\right) q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m} (q^l)_x], \tag{61}$$

so that the ODE given by (3) reads:

$$a(l+n)\phi^{2m+n}(x)[(l+n-1)\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + b_3\phi^{4m+4}(x) + b_2\phi^{4m+2}(x) + b_1 - l\omega\phi^{2m+2}(x) = 0. \tag{62}$$

By implementing the translational Lie symmetry that is supported by the above ODE (62), one can obtain the implicit quiescent optical solitons to the governing model (61) as given by (32) where

$$A = 16al^5\phi^{2m+n} + 16al^4m\phi^{2m+n} + 48al^4n\phi^{2m+n} - 16al^3m^2\phi^{2m+n} + 56al^3n^2\phi^{2m+n} - 16al^3\phi^{2m+n} - 32al^3m\phi^{2m+n} + 40al^3mn\phi^{2m+n} - 16al^2m^3\phi^{2m+n} - 32al^2m^2\phi^{2m+n} - 32al^2m^2n\phi^{2m+n} + 32al^2n^3\phi^{2m+n} + 36al^2mn^2\phi^{2m+n} - 16al^2m\phi^{2m+n} - 32al^2n\phi^{2m+n} - 64al^2mn\phi^{2m+n} - 24alm^3n\phi^{2m+n} - 20alm^2n^2\phi^{2m+n} - 48alm^2n\phi^{2m+n} + 9aln^4\phi^{2m+n} + 14almn^3\phi^{2m+n} - 20aln^2\phi^{2m+n} - 40almn^2\phi^{2m+n} - 24almn\phi^{2m+n} - 8am^3n^2\phi^{2m+n} - 4am^2n^3\phi^{2m+n} - 16am^2n^2\phi^{2m+n} + an^5\phi^{2m+n} + 2amn^4\phi^{2m+n} - 4an^3\phi^{2m+n} - 8amn^3\phi^{2m+n} - 8amn^2\phi^{2m+n} \tag{63}$$

$$B = 16b_2l^3\phi^{4m+2} + 16b_3l^3\phi^{4m+4} + 16b_1l^3 + 24b_2l^2n\phi^{4m+2} + 24b_3l^2n\phi^{4m+4} - 16b_3l^2\phi^{4m+4} + 32b_1l^2m + 24b_1l^2n + 16b_1l^2 - 16b_2lm^2\phi^{4m+2} - 16b_3lm^2\phi^{4m+4} + 16b_1lm^2 + 12b_2ln^2\phi^{4m+2} + 12b_3ln^2\phi^{4m+4} - 16b_3ln\phi^{4m+4} + 32b_1lmn - 16b_2l\phi^{4m+2} - 32b_2lm\phi^{4m+2} - 16b_3lm\phi^{4m+4} + 16b_1lm + 12b_1ln^2 + 16b_1ln - 8b_2m^2n\phi^{4m+2} - 8b_3m^2n\phi^{4m+4} + 8b_1m^2n + 2b_2n^3\phi^{4m+2} + 2b_3n^3\phi^{4m+4} - 4b_3n^2\phi^{4m+4} + 8b_1mn^2 - 16b_2mn\phi^{4m+2} - 8b_2n\phi^{4m+2} - 8b_3mn\phi^{4m+4} + 8b_1mn + 2b_1n^3 + 4b_1n^2 - 16l^4\omega\phi^{2m+2} - 24l^3n\omega\phi^{2m+2} - 16l^3m\omega\phi^{2m+2} + 16l^2m^2\omega\phi^{2m+2} - 12l^2n^2\omega\phi^{2m+2} - 16l^2mn\omega\phi^{2m+2} + 16l^2\omega\phi^{2m+2} + 32l^2m\omega\phi^{2m+2} + 16lm^3\omega\phi^{2m+2} + 8lm^2n\omega\phi^{2m+2} + 32lm^2\omega\phi^{2m+2} - 2ln^3\omega\phi^{2m+2} - 4lmn^2\omega\phi^{2m+2} + 8ln\omega\phi^{2m+2} + 16lmn\omega\phi^{2m+2} + 16lm\omega\phi^{2m+2}, \tag{64}$$

provided the parameter constraint given by (35) remain valid. This guarantees the existence of the quiescent solitons.

### 3.11. Quadratic-cubic law

For quadratic-cubic law of SPM, one writes:

$$F(|q|^2) = b_1|q| + b_2|q|^2, \tag{65}$$

for constants $b_1$ and $b_2$. Thus, the governing perturbed NLSE is of the following form:

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q| + b_2|q|^2)q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1 (|q|^{2m})_x q^l + \theta_2 |q|^{2m} (q^l)_x]. \tag{66}$$

while the ODE (3) simplifies to

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q| + b_2|q|^2)q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (67)$$

Using translational Lie symmetry, this ODE integrates to the implicit quiescent optical soliton solution in terms of Appell's hypergeometric function as

$$x = \pm\frac{1}{n}\sqrt{\frac{2a(l+n)(2l+n)\phi^n}{l\omega}} \times F_1\left(\frac{n}{2};\frac{1}{2},\frac{1}{2};\frac{2+n}{2};-\frac{2\phi b_2 A_1}{b_1 A_2 + A_3},-\frac{2\phi b_2 A_1}{b_1 A_2 - A_3}\right), \quad (68)$$

where

$$A_1 = (2l+n)(2l+n+1), \quad (69)$$

$$A_2 = (2l+n)(2l+n+2), \quad (70)$$

and

$$A_3 = \sqrt{A_2\{A_2 b_1^2 + 4b_2 l\omega(2l+n+1)^2\}}. \quad (71)$$

The constraint condition (26) in this case implies

$$\max\left(\left|\frac{2\phi b_2 A_1}{b_1 A_2 + A_3}\right|, \left|\frac{2\phi b_2 A_1}{b_1 A_2 - A_3}\right|\right) < 1. \quad (72)$$

along with

$$b_1 A_2 \pm A_3 \neq 0, \quad (73)$$

which together guarantees the existence of quiescent optical solitons.

### 3.12. Genearlized quadratic-cubic law

For the generalized quadratic–cubic law of nonlinearity, the SPM structure is

$$F(|q|^2) = b_1|q|^m + b_2|q|^{m+1}, \quad (74)$$

for non-zero constants $b_1$ and $b_2$. For $m = 1$, (74) falls back to the SPM structure of quadratic-cubic law, namely (65). Therefore, the perturbed NLSE for this SPM structure is given by

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^m + b_2|q|^{m+1})q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (75)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \{\phi'(x)\}^2 + \phi^2(x)\{b_1\phi^m(x) + b_2\phi^{m+1}(x) - l\omega\} = 0. \quad (76)$$

Using the translational invariance that is supported by the above ODE, one arrives at the implicit quiescent optical soliton given by (32), along with the parameter constraints (35), where

$$A = a(l+n)(2l+n)(2l+m+n) \times (2l+m+n+1)\phi^{n-2}, \quad (77)$$

and

$$B = 2\begin{bmatrix}(2l+n)\phi^m\{b_2\phi(2l+m+n) + \\ +b_1(2l+m+n+1)\} - \\ -l\omega(2l+m+n)(2l+m+n+1)\end{bmatrix}. \quad (78)$$

### 3.13. Quadratic-cubic-quartic law

For quadratic–cubic–quartic law of refractive index change, the SPM structure is

$$F(|q|^2) = b_1|q| + b_2|q|^2 + b_3|q|^3, \quad (79)$$

for non-zero constants $b_j$ with $j = 1, 2, 3$. The corresponding perturbed NLSE is

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q| + b_2|q|^2 + b_3|q|^3)q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (80)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x)\{\phi'(x)\}^2 + \phi^2(x)\{b_1\phi(x) + b_2\phi^2(x) + b_3\phi^3(x) - l\omega\} = 0. \quad (81)$$

Using the translational Lie symmetry supported by the ODE (81), one recovers the implicit quiescent optical solitons given by (32), along with the constraint (35), where

$$A = 16al^5\phi^{n-2} + 48al^4\phi^{n-2} + 48al^4n\phi^{n-2} + 56al^3n^2\phi^{n-2} + 44al^3\phi^{n-2} + 120al^3n\phi^{n-2} + 32al^2n^3\phi^{n-2} + 108al^2n^2\phi^{n-2} + 12al^2\phi^{n-2} + 88al^2n\phi^{n-2} + 9aln^4\phi^{n-2} + 42aln^3\phi^{n-2} + 55aln^2\phi^{n-2} + 18aln\phi^{n-2} + an^5\phi^{n-2} + 6an^4\phi^{n-2} + 11an^3\phi^{n-2} + 6an^2\phi^{n-2}, \quad (82)$$

and

$$B = 16b_3l^3\phi^3 + 16b_2l^3\phi^2 + 16b_1l^3\phi + 24b_3l^2n\phi^3 + 24b_2l^2n\phi^2 + 24b_1l^2n\phi + 24b_3l^2\phi^3 + 32b_2l^2\phi^2 + 40b_1l^2\phi + 12b_3ln^2\phi^3 + 12b_2ln^2\phi^2 + 12b_1ln^2\phi + 24b_3ln\phi^3 + 32b_2ln\phi^2 + 40b_1ln\phi + 8b_3l\phi^3 + 12b_2l\phi^2 + 24b_1l\phi + 2b_3n^3\phi^3 + 2b_2n^3\phi^2 + 2b_1n^3\phi + 6b_3n^2\phi^3 + 8b_2n^2\phi^2 + 10b_1n^2\phi + 4b_3n\phi^3 + 6b_2n\phi^2 + 12b_1n\phi - 16l^4\omega - 24l^3n\omega - 48l^3\omega - 12l^2n^2\omega - 48l^2n\omega - 44l^2\omega - 2ln^3\omega - 12ln^2\omega - 22ln\omega - 12l\omega. \quad (83)$$

### 3.14. Generalized quadratic-cubic-quartic law

The nonlinearity structure of SPM is

$$F(|q|^2) = b_1|q|^m + b_2|q|^{m+1} + b_3|q|^{2m+1}, \quad (84)$$

which means the perturbed NLSE is given by

$$i(q^l)_t + a(|q|^n q^l)_{xx} + (b_1|q|^m + b_2|q|^{m+1} + b_3|q|^{2m+1})q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \quad (85)$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n) \times \phi^n(x)\{\phi'(x)\}^2 + \phi^2(x) \times \{b_1\phi^m(x) + b_2\phi^{m+1}(x) + b_3\phi^{2m+1}(x) - l\omega\} = 0. \quad (86)$$

The translational Lie symmetry when implemented leads to the quiescent optical solitons as given by (32), together with the constraints for the existence of these solitons as given by (35), where in this case:

$$A = 16al^5\phi^{n-2} + 32al^4m\phi^{n-2} + 16al^4\phi^{n-2} + 48al^4n\phi^{n-2} + 20al^3m^2\phi^{n-2} + 20al^3m\phi^{n-2} + 80al^3mn\phi^{n-2} + 56al^3n^2\phi^{n-2} + 4al^3\phi^{n-2} + 40al^3n\phi^{n-2} + 4al^2m^3\phi^{n-2} + 6al^2m^2\phi^{n-2} + 40al^2m^2n\phi^{n-2} + 72al^2mn^2\phi^{n-2} + 2al^2m\phi^{n-2} + 40al^2mn\phi^{n-2} + 32al^2n^3\phi^{n-2} + 36al^2n^2\phi^{n-2} + 8al^2n\phi^{n-2} + 6alm^3n\phi^{n-2} + 25alm^2n^2\phi^{n-2} + 9alm^2n\phi^{n-2} + 28almn^3\phi^{n-2} + 25almn^2\phi^{n-2} + 3almn\phi^{n-2} + 9aln^4\phi^{n-2} + 14aln^3\phi^{n-2} + 5aln^2\phi^{n-2} + 2am^3n^2\phi^{n-2} + 5am^2n^3\phi^{n-2} + 3am^2n^2\phi^{n-2} + 4amn^4\phi^{n-2} + 5amn^3\phi^{n-2} + amn^2\phi^{n-2} + an^5\phi^{n-2} + 2an^4\phi^{n-2} + an^3\phi^{n-2} \tag{87}$$

and

$$B = 16b_2l^3\phi^{m+1} + 16b_3l^3\phi^{2m+1} + 16b_1l^3\phi^m + 24b_2l^2n\phi^{m+1} + 24b_3l^2n\phi^{2m+1} + 24b_1l^2n\phi^m + 8b_2l^2\phi^{m+1} + 24b_2l^2m\phi^{m+1} + 8b_3l^2\phi^{2m+1} + 16b_3l^2m\phi^{2m+1} + 16b_1l^2\phi^m + 24b_1l^2m\phi^m + 8b_2lm^2\phi^{m+1} + 4b_3lm^2\phi^{2m+1} + 8b_1lm^2\phi^m + 12b_2ln^2\phi^{m+1} + 12b_3ln^2\phi^{2m+1} + 12b_1ln^2\phi^m + 8b_2ln\phi^{m+1} + 24b_2lmn\phi^{m+1} + 8b_3ln\phi^{2m+1} + 16b_3lmn\phi^{2m+1} + 16b_1ln\phi^m + 24b_1lmn\phi^m + 4b_2lm\phi^{m+1} + 4b_3lm\phi^{2m+1} + 4b_1l\phi^m + 12b_1lm\phi^m + 4b_2m^2n\phi^{m+1} + 2b_3m^2n\phi^{2m+1} + 4b_1m^2n\phi^m + 2b_2n^3\phi^{m+1} + 2b_3n^3\phi^{2m+1} + 2b_1n^3\phi^m + 6b_2mn^2\phi^{m+1} + 2b_2n^2\phi^{m+1} + 4b_3mn^2\phi^{2m+1} + 2b_3n^2\phi^{2m+1} + 6b_1mn^2\phi^m + 4b_1n^2\phi^m + 2b_2mn\phi^{m+1} + 2b_3mn\phi^{2m+1} + 6b_1mn\phi^m + 2b_1n\phi^m - 16l^4\omega - 32l^3m\omega - 24l^3n\omega - 16l^3\omega - 20l^2m^2\omega - 32l^2mn\omega - 20l^2m\omega - 12l^2n^2\omega - 16l^2n\omega - 4l^2\omega - 4lm^3\omega - 10lm^2n\omega - 6lm^2\omega - 8lmn^2\omega - 10lmn\omega - 2lm\omega - 2ln^3\omega - 4ln^2\omega - 2ln\omega. \tag{88}$$

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### 3.15. Parabolic non-local law

This law of noinlinearity has its functional $F$ of the following form:

$$F(|q|^2) = b_1|q|^2 + b_2|q|^4 + b_3(|q|^2)_{xx}, \tag{89}$$

for real-valued constants $b_j$ with $j$ =1, 2, 3. Therefore, the governing NLSE, along with its perturbation terms, is

$$i(q^l)_t + a(|q|^nq^l)_{xx} + [b_1|q|2 + b_2|q|^4 + b_3(|q|^2)_{xx}]q^l = i[\lambda(|q|^{2m}q^l)_x + \theta_1(|q|^{2m})_xq^l + \theta_2|q|^{2m}(q^l)_x], \tag{90}$$

while the ODE (3) simplifies to:

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\times \phi^n(x)\{\phi'(x)\}^2 + \phi^2(x)\times \left[\begin{matrix} b_1\phi^2(x) + b_2\phi^4(x) + \\ 2b_3\{\phi(x)\phi''(x) + \{\phi'(x)\}^2\} - l\omega \end{matrix}\right] = 0. \tag{91}$$

For integrability, we must choose

$$l + n = 0. \tag{92}$$

This reduces equations (90) and (91), respectively, to

$$\frac{i}{(q^n)_t} + a\left(\frac{|q|^n}{q^n}\right)_{xx} + \frac{b_1|q|2 + b_2|q|^4 + b_3(|q|^2)_{xx}}{q^n} = i\left[\lambda\left(\frac{|q|^{2m}}{q^n}\right)_x + \theta_1\frac{(|q|^{2m})_x}{q^n} + \theta_2\frac{|q|^{2m}}{(q^n)_x}\right], \tag{93}$$

and

$$b_1\phi^2(x) + b_2\phi^4(x) + 2b_3\{\phi(x)\phi''(x) + \{\phi'(x)\}^2\} - -l\omega = 0. \tag{94}$$

Here in (93), the relation between the perturbation parameters $\lambda$ and $\theta_j$ for $j$ = 1, 2 is indicated in (4). Upon integration one recovers the implicit solution to (94) as

$$x = \pm\sqrt{\frac{24b_3}{3b_1 - \sqrt{9b_1^2 - 48n\omega b_2}}}\times F\left(i\sinh^{-1}\left(2\phi\sqrt{\frac{b_2}{3b_1 + \sqrt{9b_1^2 - 48n\omega b_2}}}\right) \middle| \frac{3b_1 + \sqrt{9b_1^2 - 48n\omega b_2}}{3b_1 - \sqrt{9b_1^2 - 48n\omega b_2}}\right) \tag{95}$$

where $F(\psi|m)$ is elliptic integral of the first kind and is defined as follows:

$$F(\psi|m) = \int_0^\psi \frac{1}{\sqrt{1 - m\sin^2\theta}}d\theta, \tag{96}$$

with

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2}, \tag{97}$$

and

$$m\sin^2\psi < 1. \tag{98}$$

The additional parameteric restrictions that stem from (95) are

$$3b_1^2 - 16n\omega b_2 > 0, \tag{99}$$

$$b_2\left(3b_1 + \sqrt{9b_1^2 - 48n\omega b_2}\right) > 0, \tag{100}$$

and

$$b_3\left(3b_1 - \sqrt{9b_1^2 - 48n\omega b_2}\right) > 0, \tag{101}$$

which must hold for the quiescent solitons to exist.

### 3.16. LOG-law

For log-law nonlinear refractive index, the SPM is given in the form:

$$F(|q|^2) = b\ln|q|^2, \tag{102}$$

for non-zero constant $b$. In this case the governing model is given by

$$i(q^l)_t + a(|q|^n q^l)_{xx} + b(\ln|q|^2)q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x], \tag{103}$$

while the ODE (3) simplifies to

$$a(l+n)\phi^{n+1}(x)\phi''(x) + a(l+n-1)(l+n)\phi^n(x) \times \{\phi'(x)\}^2 + \phi^2(x)\{2b\ln\phi(x) - l\omega\} = 0. \tag{104}$$

From the translational Lie symmetry that is supported by the above ODE one recovers the quiescent optical soliton solution as

$$x = \pm\phi^{\frac{n}{2}}\sqrt{-\frac{2a(l+n)(2l+n)}{bn}} \times G\left(\frac{\sqrt{n\{(2l+n)(2b\ln\phi - \omega l) - 2b\}}}{2\sqrt{b(2l+n)}}\right), \tag{105}$$

where the Dawson integral is defined by

$$G(x) = e^{-x^2}\int_0^x e^{y^2}\,dy. \tag{106}$$

Another constraint that follows from (105), for the existence of solitons is given by

$$ab < 0. \tag{107}$$

### 3.17. Exponential law

For exponential law of nonlinearity, the functional $F$ reads:

$$F(|q|^2) = \frac{1}{b}\left(1 - e^{-b|q|^2}\right), \tag{108}$$

for non-zero $b$. In this case the perturbed NLSE is given by

$$i(q^l)_t + a(|q|^n q^l)_{xx} + \frac{1}{b}\left(1 - e^{-b|q|^2}\right)q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x]. \tag{109}$$

The corresponding ODE takes the form:

$$ab(l+n)e^{b\phi^2(x)}\phi^n(x) \times [(l+n-1)\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + \phi^2(x)\{(1 - bl\omega)e^{b\phi^2(x)} - 1\}, \tag{110}$$

which, by virtue of the translational Lie symmetry, integrates to

$$x = \pm\int\sqrt{\frac{ab(l+n)(2l+n)\phi^{n-2}}{2(bl\omega - 1) - (2l+n)E_{-l-\frac{n}{2}+1}(b\phi^2)}}\,d\phi, \tag{111}$$

with the exponential integral being defined as

$$E_m(z) = \int_1^\infty \frac{e^{t(-z)}}{t^m}\,dt. \tag{112}$$

### 3.18. Saturating law

For saturating law of nonlinearity, the nonlinear functional $F$ takes the form:

$$F(|q|^2) = \frac{b_1|q|^2}{b_2 + b_3|q|^2}, \tag{113}$$

for non-zero constants $b_j$ with $j$ =1, 2, 3. The perturbed NLSE is

$$i(q^l)_t + a(|q|^n q^l)_{xx} + \frac{b_1|q|^2}{b_2 + b_3|q|^2}q^l = i[\lambda(|q|^{2m} q^l)_x + \theta_1(|q|^{2m})_x q^l + \theta_2|q|^{2m}(q^l)_x]. \tag{114}$$

The corresponding ODE for $\phi(x)$ is

$$a(l+n)\phi^n(x)[(l+n-1)\{\phi'(x)\}^2 + \phi(x)\phi''(x)] + \phi^2(x)\left\{\frac{b_1\phi^2(x)}{b_3\phi^2(x) + b_2} - l\omega\right\} = 0. \tag{115}$$

Upon applying the translational symmetry to the above ODE (115), one recovers the implicit quiescent optical soliton as

$$x = \pm\int\frac{1}{\phi}\sqrt{\frac{ab_3(l+n)(2l+n)\phi^n}{2\left[\left\{{}_2F_1\left(1, l+\frac{n}{2}; 1+l+\frac{n}{2}; -\frac{\phi^2 b_3}{b_2}\right) - 1\right\}b_1 + l\omega b_3\right]}}\,d\phi. \tag{116}$$

Finally, the constraint condition from the Gauss hypergeometric function gives

$$-\sqrt{\frac{b_2}{b_3}} < \phi(x) < \sqrt{\frac{b_2}{b_3}}, \tag{117}$$

for the quiescent optical solitons to exist. Another parametric constraint that comes out of (117) is given by

$$b_2 b_3 > 0. \tag{118}$$

### 4. Conclusions

This paper derived the implicit quiescent perturbed optical solitons for the NLSE with perturbation terms which appeared with arbitrary intensity. The CD was rendered to be nonlinear, while the temporal evolution was generalized from its linear counterpart. The ascertained results are a generalized version of the previously reported work, where the same model was addressed with linear temporal evolution [1]. Therefore, the results of this paper collapses to the ones in the previously reported work. Also, there are eighteen forms of SPM structures that were handled. The integration algorithm adopted in this work is Lie symmetry. The solitons solutions are all implicit and mostly appeared in terms of quadratures. Many of the solutions are in terms of a wide range of special functions. These are the

Gauss hypergeometric functions, Appell hypergeometric function, Dawson's integral, elliptic integral of the first kind and exponential integral. The results, nevertheless, stand on a strong footing to proceed further along.

The novelty of the current approach stems from its uniqueness. The Lie symmetry approach is a unique and one of the most powerful mathematical approaches that handles the recovered ODEs for every single form of SPM structure in its most unique form. The translational symmetry yielded solutions that are not recoverable by any additional approaches. Additional approaches that have been studied earlier to recover quiescent optical solitons implemented, namely the extended Jacobi's elliptic function approach, extended trial function approach, methods of first integrals, sine-Gordon equation procedure, *F*-expansion method, Riccati equation expansion methodology, Kudryashov's approach, *G'*/*G*-expansion scheme do not reveal the unique structure of the solutions that are recoverable only by the usage of Lie symmetry. This makes the paper unique and consequently sheds light on the novelty of the work.

The model can be modified and applied to various optoelectronic devices. A few these devices that would be immediately addressed are optical couplers, optical metamaterials, magneto-optic waveguides, Bragg gratings, fibers with differential group delay and dispersion-flattened fibers as well. The Lie symmetry approach would be implemented for each of these devices that would lead to the emergence of the quiescent optical solitons, explicit or implicit and/or in terms of quadratures or closed form solutions. The variety of results would be revealed with time and would be disseminated across the board after the results are all aligned with the pre-existing ones.

## Disclosure

The authors claim there us no conflict of interest in this work.

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**Неявне стаціонарне оптичне солітонне збурення з нелінійною хроматичною дисперсією та узагальненою часовою еволюцією, що має безліч структур фазової самодуляції за симетрією Лі**

**A.R. Adem, A. Biswas & Y. Yildirim**

**Анотація.** У цій статті відновлено стаціонарні збурені оптичні солітони, що виникають з нелінійного рівняння Шредінгера з гамільтоніанами збурень довільної інтенсивності. Модель розглядається з узагальненою часовою еволюцією та нелінійною хроматичною дисперсією. Враховується вісімнадцять форм структур фазової самомодуляції. Інтегрування здійснюється застосуванням симетрії Лі. Також представлено параметричні обмеження, які гарантують існування таких солітонів.

**Ключові слова:** нелінійна дисперсія, стаціонарні солітони.