Optics

Cubic-quartic optical soliton perturbation with Fokas–Lenells equation by semi-inverse variation

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Abstract. This paper recovers cubic-quartic bright optical solitons with perturbed Fokas–Lenells equation. The Hamiltonian perturbation terms appear with maximal permissible intensity. The semi-inverse variational principle is employed to retrieve such solitons.

Keywords: soliton, cubic-quartic, perturbation, Fokas–Lenells equation.

https://doi.org/10.15407/spqeo24.04.431
PACS 42.65.Tg, 42.81.Dp

Manuscript received 16.06.21; revised version received 07.09.21; accepted for publication 10.11.21; published online 23.11.21.

1. Introduction

One of the modern trends to address soliton dynamics is the implementation of cubic-quartic (CQ) dispersion profile that replaces the traditional chromatic dispersion (CD), when it runs low. There are various other aspects that circumvents this crisis. A few of them replacing the governing nonlinear Schrödinger's equation by Schrödinger–Hirota equation, introduction to Bragg gratings, work with pure quartic solitons or pure cubic solitons and so on [1–20]. This paper implements one of the latest ways to address the low count on CD, namely: as introduced, CQ solitons by virtue of another model that was recently introduced in the literature. This is Fokas–Lenells equation (FLE), which will be studied with CQ dispersion profile and will be referred to as CQ-FLE. It must be noted that some preliminary results with CQ-FLE have been reported [11]. Technology of adopting CQ dispersion profile to replace the low count of CD is not new. In fact, another model from quantum optics, namely Lakshmanan–Porsezian–Daniel model with CQ dispersion profile, has been lately studied [1, 2].

Today's paper will address the perturbed version of CQ-FLE, where the perturbation terms are of Hamiltonian type and are studied with maximally permissible intensity that is also known as full nonlinearity. These perturbations are with self-steepening nonlinearity as well as nonlinear dispersion. Thus, the perturbed CQ-FLE will be studied by the application of semi-inverse variational principle (SVP) that will retrieve a single bright soliton solution in presence of an arbitrary full nonlinearity parameter [1–10, 12]. It must be noted that there exists a wide range of integration schemes that can retrieve soliton solutions to models with Hamiltonian perturbations, such as inverse scattering transform or Hirota's bilinear structure, but all of these approaches fail, when an arbitrary power law nonlinearity parameter is present. It is only SVP that comes to the rescue in such a situation. The bright soliton solution retrieved in this manner is, however, not an exact solution, although it is analytical, since it is based on SVP that will be introduced in the paper. The details are enumerated and exhibited in the rest of the paper.

1.1. Governing model

The unperturbed version of CQ-FLE is given as:

$$iq_t + iaq_{xxx} + bq_{xxxx} + |q|^2\left(cq + idq_x\right) = 0\,. \tag{1}$$

In (1), the dependent variable $q(x,t)$ is the complex-valued wave function that represents the soliton profile. The independent variables x and t represent spatial and temporal variables, respectively. The first term is the linear temporal evolution where $i=\sqrt{-1}$. The coefficients of a and b are 3OD and 4OD, respectively. The coefficient c is Kerr law of nonlinearity, while the coefficient d is nonlinear dispersion that compensates for low count of CD as introduced by Fokas and Lenells. Thus, the current model additionally introduced 3OD and 4OD terms that together with nonlinear dispersion collectively balance with SPM, due to Kerr law for solitons to sustain. This model has been recently studied, and its full soliton solutions spectrum has been recovered and reported [11]. The bright single-soliton solution to (1) is identified as:

$$q(x,t) = A\,\mathrm{sech}^2\left[B(x-vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{2}$$

where A is the soliton amplitude and B – inverse width with v being its velocity. From the phase component, κ is the soliton frequency, while ω – soliton wave number, and finally θ_0 – phase constant.

When perturbation terms are turned on, the corresponding perturbed CQ-FLE is given by:

$$iq_t + iaq_{xxx} + bq_{xxxx} + |q|^2\left(cq + idq_x\right) = \\ = i\left[\lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \theta|q|^{2m}q_x\right]. \tag{3}$$

The coefficients of λ, μ and θ stem from self-steepening effects and nonlinear dispersions, respectively. The parameter m represents the full nonlinearity effect that is otherwise referred to as maximum intensity as permitted by the model experimentally.

2. Mathematical preliminaries

To obtain a bright single-soliton solution to the perturbed CQ-FLE with nonlinear perturbation terms, we decompose

$$q(x,t) = g(s)e^{i\varphi(x,t)}, \tag{4}$$

where

$$s = x - vt \text{ and } \varphi(x,t) = -\kappa x + \omega t + \theta_0\,, \tag{5}$$

with $\varphi(x,t)$ being the phase component of soliton.

After inserting (4) and (5) into (3) and breaking into real and imaginary components, it leads to a pair of relations. The imaginary component is as follows:

$$(a-4b\kappa)g'' - \left(v + 3a\kappa^2 - 4b\kappa^3\right)g' + \\ + \left[d - (2m+1)\lambda - 2m\mu - \theta\right]g^2 g' = 0. \tag{5}$$

Upon setting the coefficients of linearly independent functions to zero, one recovers the parameter constraints as

$$a - 4b\kappa \tag{7}$$

and

$$d - (2m+1)\lambda - 2m\mu - \theta = 0\,, \tag{8}$$

while the velocity of soliton comes out as

$$v = -3a\kappa^2 + 4b\kappa^3. \tag{9}$$

The real part shapes up as

$$bg^{(iv)} + 3\kappa(a-2b\kappa)g'' - \left(\omega + a\kappa^3 - b\kappa^4\right)g + \\ + (c + d\kappa)g^3 - \kappa(\lambda+\theta)g^{2m+1} = 0. \tag{10}$$

Multiplying (10) by g' and integrating once leads to

$$2b(g'')^2 - 6\kappa(a-2b\kappa)(g')^2 + 2\left(\omega + a\kappa^3 - b\kappa^4\right)g^2 - \\ - (c+d\kappa)g^4 - \frac{2\kappa}{m+1}(\lambda+\theta)g^{2m+2} = K\,, \tag{11}$$

where K is the integration constant.

3. Application of SVP

From (11), the stationary integral is constructed as

$$J = \int_{-\infty}^{\infty} K ds = \\ = \int_{-\infty}^{\infty}\left\{\begin{matrix} 2b(g'')^2 - 6\kappa(a-2b\kappa)(g')^2 + \\ + 2\left(\omega + a\kappa^3 - b\kappa^4\right)g^2 - \\ -(c+d\kappa)g^4 - \dfrac{2\kappa}{m+1}(\lambda+\theta)g^{2m+2}\end{matrix}\right\}ds. \tag{12}$$

SVP states that the bright single-soliton solution to the perturbed CQ-FLE, given by (3), would be the same as that of its unperturbed version (2) [1–10]. However, the amplitude and inverse width of the perturbed soliton would vary according to [1–10]:

$$\frac{\partial J}{\partial A} = 0 \tag{13}$$

and

$$\frac{\partial J}{\partial B} = 0\,. \tag{14}$$

Therefore, substituting the unperturbed single-soliton solution (2) into (12) reduces the stationary integral to

$$J = \frac{128b}{21}A^2B^3 - 32A^2B\kappa(a-2b\kappa) + \frac{8A^2}{3B}\left(\omega + a\kappa^3 - b\kappa^4\right) - \\ - \frac{32A^4}{35B}(c+d\kappa) - \frac{16A^{2m+2}}{B}\kappa(\lambda+\theta)G, \tag{15}$$

where

$$G = \frac{m(2m+1)\Gamma(2m)\Gamma(1/2)}{(m+1)(4m+1)(4m+3)\Gamma(2m+1/2)}. \quad (16)$$

Therefore, the equations (13) and (14) are reduced to

$$80bB^4 - 420B^2\kappa(a-2b\kappa) + 35(\omega + a\kappa^3 - b\kappa^4) - \\ -24A^2(c+d\kappa) - 210(m+1)A^{2m}\kappa(\lambda+\theta)G = 0 \quad (17)$$

and

$$240bB^4 - 420B^2\kappa(a-2b\kappa) - 35(\omega + a\kappa^3 - b\kappa^4) - \\ -12A^2(c+d\kappa) + 210A^{2m}\kappa(\lambda+\theta)G = 0, \quad (18)$$

respectively. Upon uncoupling (17) and (18) leads to the quartic equation for the soliton width B as

$$PB^4 - QB^2 + R = 0, \quad (19)$$

where

$$P = 320b, \quad (20)$$

$$Q = 840\kappa(a-2b\kappa), \quad (21)$$

and

$$R = -12A^2(c+d\kappa) - 210mA^{2m}\kappa(\lambda+\theta)G. \quad (22)$$

This solves (19) to

$$B = \left[\frac{Q + \sqrt{Q^2 - 4PR}}{2P}\right]^{1/2}, \quad (23)$$

provided

$$|Q| \geq 2\sqrt{PR} \quad (24)$$

and

$$P\left(Q + \sqrt{Q^2 - 4PR}\right) > 0. \quad (25)$$

Hence, finally, the bright single-soliton solution to (3) is still given by (2), where the amplitude-width relation is dictated by (23), while the velocity of perturbed soliton is indicated by (9). This bright soliton solution can be implemented in telecommunications technology for soliton transmission across intercontinental distances. It must be noted that the perturbed FLE does not permit exact single-soliton solution by the aid of any known algorithm unless a specific value of the full nonlinearity parameter is selected. However, it is SVP can circumvent this problem although this analytical bright single-soliton solution is not exact.

4. Conclusions

This paper retrieved a bright single-soliton solution to the perturbed CQ-FLE, where CD is replaced by 3OD and 4OD. This analytical soliton solution is not exact, since the solution is retrieved as based on SVP. This method is thus a savior in the sense that an analytical soliton solutions when the full nonlinearity parameter is just arbitrary. However, this approach also has its own limitations. For example, it fails to retrieve dark or singular soliton solutions to the model when the perturbation terms carry an arbitrary power–law parameter. The stationary integral is rendered to be divergent for such cases. Nevertheless, this method seems to be promising for implementation of other forms of bright solitons such as cosh-Gaussian pulses and others. These results will be revealed in future works. Additional models, namely, those with several forms of Kudryashov's law of refractive index, will be worked upon and those results will also appear with time [13–20].

Conflict of interests

The authors declare that there is no conflict of interests.

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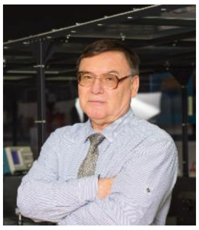

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Кубічно-квартичне оптичне збурення солітона за рівнянням Фокаса–Ленеллса напівоберненою варіацією

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Анотація. У цій роботі описано кубічно-квартичні яскраві оптичні солітони збуреним рівнянням Фокаса–Ленеллса. Члени Гамільтонових збурень з’являються з максимально припустимою інтенсивністю. Для отримання формули для таких солітонів використовується напівобернений варіаційний принцип.

Ключові слова: солітони, кубічно-квартичні, збурення, рівняння Фокаса–Ленеллса.