Optics

Optical soliton perturbation for the concatenation model with multiplicative white noise by the improved modified extended tanh–function approach

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Abstract. This paper recovers single and straddled optical solitons for the concatenation model in the presence of multiplicative white noise by using an improved modified extended tanh function approach. The parameter constraints for the existence and sustainment of such solitons are also presented. It is observed that the effect of white noise is confined to the phase component of the recovered solitons.

Keywords: straddled optical solitons, integrability, concatenation model.

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1. Introduction

In 2014, a team of scientists headed by Prof. Nail Akhmediev from the Australian National University in Canberra, Australia, proposed a model for the propagation of solitons through optical fibers. This model is a conjunction of three well-known pre-existing equations, namely the nonlinear Schrödinger equation (NLSE), Lakshmanan–Porsezian–Daniel (LPD) equation, and the Sasa–Satsuma equation (SSE) [1–3]. This newly established model was later rebranded as the concatenation model since it is formulated by concatenating three fundamental equations from fiber optics that govern the propagation of solitons through the optical fibers. Later, this model was supplemented with its corresponding dispersive version, which is labeled as the dispersive concatenation model [4, 5].

This concatenation model was extensively studied in various contexts in the field of fiber optics. The 1-soliton solution was established by the method of undetermined coefficients, and the conservation laws were located. The model was also addressed with fractional temporal evolution. The quiescent optical solitons were also recovered for nonlinear chromatic dispersion (CD) and Kerr law of self-phase modulation (SPM) by Lie symmetry. This analysis was conducted for linear temporal evolution as well as generalized temporal evolution. The model was also investigated with spatio-temporal dispersion in addition to CD. In this context the mathematical engineering of Internet bottleneck control was proposed [9]. This concatenation model was later also studied numerically by the Laplace–Adomian decomposition scheme. Subsequently, the concatenation model was addressed with the power law of SPM, where all of the above-mentioned features were successfully addressed, and a plethora of similar results have been presented. As an alternative scenario, the concatenation models were later handled in the absence of SPM.

Later, the model was applied to polarization–preserving fibers, where the soliton solutions were retrieved by the method of undetermined coefficients and other approaches. This model was later applied to study solitons with Bragg gratings. The application to magneto-optic waveguides and optical couplers was also taken into consideration. The plethora of results thus gave way to a complete picture of the concatenation model with Kerr-law and power-law forms of SPM. There are a wide range of aspects that are yet to be covered for the concatenation model as well as the dispersive concatenation model. One aspect is the study of the model in the presence of multiplicative white noise. This paper does exactly that. The concatenation model with Kerr law of SPM in the presence of white noise is taken up here. The integration algorithm that is adopted in the paper is the improved modified extended tanh–function approach [6–9]. The study of a wide range of models from a wide variety of engineering and applications from other physical sciences, with the effect of stochasticity included, was also studied [10–16]. This paper recovers a full spectrum of optical 1-soliton solution along with straddled optical solitons, which are also presented. The results are exhibited in the rest of the paper after a succinct introduction to the model and preliminary mathematical analysis.

1.1. Governing model

The stochastic perturbed concatenation model is expressed in its dimensionless form. It incorporates Kerr-law non-linearity and multiplicative white noise and is a composite of three well-known models, as described below:

$$i\psi_t + \Omega\psi_{xx} + \varrho|\psi|^2\psi + \\ +\varrho_1\begin{bmatrix}\Delta_1\psi_{xxxx} + \Delta_2(\psi_x)^2\psi^* + \Delta_3|\psi_x|^2\psi + \\ +\Delta_4|\psi|^2\psi_{xx} + \Delta_5\psi^2\psi_{xx}^* + \Delta_6|\psi|^4\psi\end{bmatrix} + \\ i\varrho_2[\Delta_7\psi_{xxx} + \Delta_8|\psi|^2\psi_x + \Delta_9\psi^2\psi_x^*] + \gamma\psi\frac{dM(t)}{dt} = \\ = i\epsilon[\psi_x + \vartheta(|\psi|^2\psi)_x + \eta(|\psi|^2)_x\psi]. \tag{1}$$

where $\psi(x,t)$ represents the soliton wave profile with x and t being the independent spatial and temporal variables respectively. The complex parameter $i = \sqrt{-1}$. The first term represents linear temporal evolution, while Ω and ϱ denote the constants for CD and the SPM, respectively. The parameter γ is the coefficient of the white noise, while $M(t)$ is the Wiener process $M(t)$, which is a continuous-time stochastic process known as Brownian motion, economics and physics for modeling random behavior. For any $0 \le s < t$, the increment $M(t) - M(s)$ is independent of the value of s and has a normal distribution with mean 0 and variance $t - s$. Additionally, for any $t > 0$, the random variable $M(t)$ has a normal distribution with mean 0 and variance t. The paths of the Wiener process are continuous, meaning it has no sudden jumps, and the probability of the path attaining any particular value over a time range is 0.

In (1), the first three terms constitute the NLSE with Kerr law of SPM. The coefficients of ϱ_1 and ϱ_2 represent the LPD model and SSE, respectively. Then, on the right-hand side of the perturbation terms, the first term gives the self-steepening effect, while the second and third terms represent soliton self-frequency shift. Apart from the first term in (1), the coefficients are all real-valued parameters. The subsequent section will carry out a preliminary analysis of the model to get started with the retrieval of the soliton solutions.

2. Mathematical preliminaries

In this section, we outline the main steps of the improved extended tanh-equation method as follows:

Suppose that we have a nonlinear evolution equation in the form:

$$\chi(\psi, \psi_t, \psi_x, \psi_{xx}, \psi_{xt}, \ldots) = 0, \tag{2}$$

where $\psi = \psi(x,t)$ is an unknown function, χ is a polynomial in ψ and its various partial derivatives ψ_t, ψ_x with respect to t, x respectively, in which the highest order derivatives and nonlinear terms are involved.

Step–1: Use the following traveling wave transformation

$$\psi(x,t) = \phi(\zeta), \quad \zeta = r(x - ct), \ c \neq 0, \tag{3}$$

where c represents the wave speed. Then, Eq. (2) can be transformed to the following nonlinear ordinary differential equation:

$$\mathrm{F}(\phi, \phi', \phi'', \phi''', \ldots) = 0. \tag{4}$$

Step–2: Suppose that the solution of Eq. (4) can be expressed in the form:

$$\phi(\zeta) = \sum_{i=0}^{N} a_i w^i + \sum_{i=1}^{N} b_i w^{-i}, \tag{5}$$

where w satisfies

$$w' = \varepsilon\sqrt{h_0 + h_1 w + h_2 w^2 + h_3 w^3 + h_4 w^4}, \tag{6}$$

where $\varepsilon = \pm 1$. The above equation introduces various kinds of fundamental solutions [6–9]. From these solutions, more new exact solutions for Eq. (2) can be obtained.

Step–3: Designate the positive integer number N in Eq. (5) by making a balance between the highest order derivatives and the nonlinear terms in Eq. (4).

Step–4: Substitute (5) into (4) along with (6). As a result of this substitution, we get a polynomial of w. In this polynomial, we gather all terms of the same powers and put them equal to zero. We get an over-determined system of algebraic equations which can be solved by the Mathematica program to get the unknown parameters r, c, a_0, a_i, and b_i ($i = 1, 2, \ldots$). Consequently, we obtain the exact solutions of (2).

3. Application to the perturbed concatenation model

In this section, we apply the improved modified extended tanh-function method to find optical soliton solutions and other solutions of Eq. (1). We suppose that the solution of this equation is

$$\psi(x,t)=\phi(\zeta)e^{i\alpha(x,t)}, \quad (7)$$

$$\zeta=r(x-ct),\ c\neq 0, \quad (8)$$

$$\alpha(x,t)=-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0, \quad (9)$$

where $\phi(x,t)$ denotes the amplitude component, τ, $\alpha(x,t)$, ς, γ and o_0 represents the frequency of the solitons, phase component, wave number, noise coefficient, phase constant respectively.

Substituting Eq. (7) into Eq. (1), we get the imaginary parts read as

$$r\phi^2\phi'\begin{pmatrix}-2(\Delta_2+\Delta_4-\Delta_5)\tau\varrho_1+\\(\Delta_8+\Delta_9)\varrho_2-2\eta-3\vartheta\end{pmatrix}- \\ -r\begin{pmatrix}\phi'(c-4\Delta_1\tau^3\varrho_1+3\Delta_7\tau^2\varrho_2+2\tau\Omega+\epsilon)\\+r^2\phi^{(3)}(4\Delta_1\tau\varrho_1-\Delta_7\varrho_2)\end{pmatrix}=0, \quad (10)$$

while the real part gives:

$$\phi^3\begin{pmatrix}\tau\big((\Delta_8-\Delta_9)\varrho_2-(\Delta_2-\Delta_3+\Delta_4+\Delta_5)\tau\varrho_1\big)-\\-\tau\vartheta+\varrho\end{pmatrix}+ \\ +\Delta_6\varrho_1\phi^5+\Delta_1r^4\varrho_1\phi^{(4)}+ \\ \phi\begin{pmatrix}\gamma^2+\Delta_1\tau^4\varrho_1-\Delta_7\tau^3\varrho_2+(\Delta_2+\Delta_3)r^2\varrho_1(\phi')^2\\-\tau(\tau\Omega+\epsilon)-\varsigma\end{pmatrix}+ \\ r^2\phi''^{(-6\Delta_1\tau^2\varrho_1+3\Delta_7\tau\varrho_2+\Omega)}+(\Delta_4+\Delta_5)r^2\varrho_1\phi^2\phi'. \quad (11)$$

From the imaginary part, we conclude that

$$c=-(-4\Delta_1\tau^3\varrho_1+3\Delta_7\tau^2\varrho_2+2\tau\Omega+\epsilon),$$

$$\Delta_7=\frac{4\Delta_1\tau\varrho_1}{\varrho_2},$$

$$\eta=\frac{1}{2}(-2(\Delta_2+\Delta_4-\Delta_5)\tau\varrho_1+(\Delta_8+\Delta_9)\varrho_2-3\vartheta). \quad (12)$$

Then, from Eq. (12) into Eq. (11), we can rewrite the real part as

$$H_5\phi^5+H_6\phi^3+H_1\phi^2\phi''+H_2\phi''+H_3\phi(\phi')^2+H_4\phi+ \\ +r^2\phi^{(4)} \quad (13)$$

with

$$100H_1=\frac{\Delta_4+\Delta_5}{\Delta_1},\quad H_2=\frac{6\Delta_1\tau^2\varrho_1+\Omega}{\Delta_1\varrho_1},\quad H_3=\frac{\Delta_2+\Delta_3}{\Delta_1},$$

$$H_4=\frac{\gamma^2-3\Delta_1\tau^4\varrho_1-\tau(\tau\Omega+\epsilon)-\varsigma}{\Delta_1r^2\varrho_1},H_5=\frac{\Delta_6}{\Delta_1r^2},$$

$$H_6=\frac{\tau\big((\Delta_8-\Delta_9)\varrho_2-(\Delta_2-\Delta_3+\Delta_4+\Delta_5)\tau\varrho_1\big)-\tau\vartheta+\varrho}{\Delta_1r^2\varrho_1}.$$

Balancing the terms $\phi^{(4)}$ and ϕ^5 yields, the balance number $N=1$. Then the solution of Eq. (13) represent by

$$\phi(\zeta)=a_0+a_1w(\zeta)+b_1\frac{1}{w(\zeta)}, \quad (14)$$

where a_0, a_1 and b_1 are constants to be determined such that $a_1\neq 0$ or $b_1\neq 0$.

Substituting the solution Eq. (14), which satisfies the condition Eq. (6), into Eq. (13) leads to a set of nonlinear equations that can be solved with the aid of Mathematica software tool. Then, the following results emerge:

Case–1: $h_0=h_1=h_3=0$,

$$a_0=0,\ a_1=\pm2\sqrt{\frac{2h_4(9H_2+10H_4)}{h_2(h_2(H_1+H_3)+H_6)}}$$

$$b_1=0, r=\pm\sqrt{\frac{-h_2H_2-H_4}{h_2}},$$

$$H_5=\frac{\begin{pmatrix}h_2(H_1+H_3)\\+H_6\end{pmatrix}\begin{pmatrix}2H_4(h_2(H_3-4H_1)+6H_6)+\\3h_2H_2(h_2(H_3-2H_1)+4H_6)\end{pmatrix}}{2(9h_2H_2+10H_4)^2}. \quad (15)$$

We can get from this case the solutions of Eq. (1) under the following conditions:

when $h_2>0$, $h_4<0$, we get a bright soliton solution as follows:

$$\psi(x,t)=\pm\sqrt{\frac{-2(9h_2H_2+10H_4)}{h_2(H_1+H_3)+H_6}}sech\left(\sqrt{h_2}r(x-ct)\right) \\ \times e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}, \quad (16)$$

when $h_2>0$, $h_4>0$, we get a singular soliton solution as follows:

$$\psi(x,t)=\pm\sqrt{\frac{2(9h_2H_2+10H_4)}{h_2(H_1+H_3)+H_6}}csch\left(\sqrt{h_2}r(x-ct)\right) \\ \times e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (17)$$

Case–2: $h_1=h_3=0, h_0=\frac{h_2^2}{4h_4}$ $, h_2<0, h_4>0$

$$a_0=0,\ a_1=\pm2\sqrt{\frac{h_4(3h_2H_2+5H_4)}{h_2(h_2(4H_1-H_3)+4H_6)}},\ b_1=0,$$

$$r=\pm\sqrt{\frac{-2H_4(h_2(H_1+H_3)+H_6)-h_2H_2(h_2(2H_1+H_3)+2H_6)}{2h_2^2(h_2(4H_1-H_3)+4H_6)}},$$

$$H_5=\frac{\begin{pmatrix}h_2(4H_1-H_3)\\+4H_6\end{pmatrix}\begin{pmatrix}6h_2H_2H_6+\\H_4(h_2(H_3-4H_1)+6H_6)\end{pmatrix}}{4(3h_2H_2+5H_4)^2}. \quad (18)$$

From this case, we can get the solutions of Eq. (1) as following:

We get a dark soliton solution as follows:

$$\psi(x,t)=\pm\sqrt{\frac{-2(3h_2H_2+5H_4)}{h_2(4H_1-H_3)+4H_6}}\tanh\left(\sqrt{\frac{-h_2}{2}}r(x-ct)\right) \\ \times e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (19)$$

We get a singular soliton solution as follows:

$$\psi(x,t)=\pm\sqrt{\frac{-2(3h_2H_2+5H_4)}{h_2(4H_1-H_3)+4H_6}}\times \\ \coth\left(\sqrt{\frac{-h_2}{2}}r(x-ct)\right)e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (20)$$

Case–3: $h_0 = h_1 = 0, h_2 > 0, h_4 > 0$

$$a_0 = b_1 = 0, \quad a_1 = \pm 2\sqrt{\frac{3h_4H_2}{3h_2H_1+2h_2H_3}},$$

$$H_5 = -\frac{h_2(12H_1^2+11H_3H_1+2H_3^2)}{60H_2}, H_4 = \frac{-4}{5}H_2h_2,$$

$$H_6 = \frac{9h_3^2H_1+6h_3^2H_3-12h_2h_4H_1-16h_2h_4H_3}{24h_4}, r = \pm\sqrt{\frac{-H_2}{5h_2}}. \quad (21)$$

From this case, we arrive at the bright-singular straddled soliton solutions of Eq. (1) as

$$\psi(x,t) = \pm 2\sqrt{\frac{3h_4H_2}{3h_2H_1+2h_2H_3}} \times \times \left[\frac{-h_2 sech^2\left(\frac{\sqrt{h_2}}{2}r(x-ct)\right)}{\pm 2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}r(x-ct)\right)+h_3}\right] \times e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}, \quad (22)$$

$$\psi(x,t) = \pm 2\sqrt{\frac{3h_4H_2}{3h_2H_1+2h_2H_3}} \times \times \left[\frac{h_2 csch^2\left(\frac{\sqrt{h_2}}{2}r(x-ct)\right)}{\pm 2\sqrt{h_2h_4}\coth\left(\frac{\sqrt{h_2}}{2}r(x-ct)\right)+h_3}\right] \times e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (23)$$

Case–4: $h_1 = h_{13} = 0, h_0 = \frac{h_2^2g^2(1-g^2)}{h_4(2g^2-1)^2}$,

$h_2 > 0, h_4 < 0$

$$a_0 = b_1 = 0, a_1 = \pm\sqrt{\frac{-h_4(6g_1h_2H_2+20g_0H_4)}{h_2(h_2(g_2H_1+g_3H_3)+g_2H_6)}},$$

$$r = \pm\sqrt{\frac{g_0H_4(h_2(H_1+H_3)+H_6) + h_2H_2(h_2(g_0H_1+g_4H_3)+g_0H_6)}{h_2^2(h_2(g_2H_1+g_3H_3)+g_2H_6)}},$$

$$H_5 = \frac{(h_2(g_2H_1+g_3H_3)+g_2H_6) \times \begin{pmatrix} 2g_0H_4(h_2(H_3-4H_1)+6H_6) + \\ 3h_2H_2\left(4g_0H_6-(g_2+2)h_2(2H_1-H_3)\right)\end{pmatrix}}{2(3g_1h_2H_2+10g_0H_4)^2} \quad (24)$$

with

$$\begin{cases} 101g_0 = (1-2g^2)^2, \quad g_1 = 16g^4-16g^2+3, \\ g_2 = 8g^4-8g^2-1, \quad g_3 = -12g^4+12g^2-1, \\ g_4 = 6g^4-6g^2+1. \end{cases}$$

We get a Jacobi elliptic solution as follows:

$$\psi(x,t) = \pm\sqrt{\frac{g^2(6g_1h_2H_2+20g_0H_4)}{-\sqrt{g_0}(h_2(g_2H_1+g_3H_3)+g_2H_6)}} \times cn\left(\sqrt{\frac{h_2}{2g^2-1}}r(x-ct)\right)e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (25)$$

and if $g \longrightarrow 1^-$, one recovers a bright soliton solution:

$$\psi(x,t) = \pm\sqrt{\frac{18h_2H_2+20H_4}{h_2(-H_1-H_3)-H_6}} \times sech\left(\sqrt{h_2}r(x-ct)\right)e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (26)$$

Case–5: $h_1 = h_3 = 0, h_0 = \frac{h_2^2(1-g^2)}{h_4(2-g^2)^2}, h_2 > 0, h_4 < 0$

$$a_0 = b_1 = 0, a_1 = \pm\sqrt{\frac{-h_4(6g_6h_2H_2+20g_5H_4)}{-h_2(h_2(g_7H_1+g_8H_3)+g_7H_6)}},$$

$$r = \pm\sqrt{\frac{-g_5H_4(h_2(H_1+H_3)+H_6) - h_2H_2(h_2(g_5H_1+g_9H_3)+g_5H_6)}{h_2^2(h_2(g_7H_1+g_8H_3)+g_7H_6)}},$$

$$H_5 = \frac{(h_2(g_7H_1+g_8H_3)+g_7H_6) \times \begin{pmatrix} 3h_2H_2(g^4h_2(H_3-2H_1)+4g_5H_6) + \\ 2g_5H_4(h_2(H_3-4H_1)+6H_6)\end{pmatrix}}{2(3g_6h_2H_2+10g_5H_4)^2} \quad (27)$$

with

$$\begin{cases} 102g_5 = (g^2-2)^2, \quad g_6 = 3g^4-8g^2+8, \\ g_7 = g^4-16g^2+16, \quad g_8 = g^4+4g^2-4, \\ g_9 = g^4-2g^2+2. \end{cases}$$

We get a Jacobi elliptic solution as follows:

$$\psi(x,t) = \pm\sqrt{\frac{g^2(6g_2h_2H_2+20g_5H_4)}{\sqrt{g_5}h_2(h_2(g_7H_1+g_8H_3)+g_7H_6)}} \times dn\left(\sqrt{\frac{h_2}{2-g^2}}r(x-ct)\right)e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (28)$$

and for $g \longrightarrow 1^-$, a bright soliton solution appears:

$$\psi(x,t) = \pm\sqrt{\frac{18h_2H_2+20H_4}{-h_2(h_2(H_1+H_3)+H_6)}} \times sech\left(\sqrt{h_2}r(x-ct)\right)e^{i(-\tau x+\varsigma t+\gamma M(t)-t\gamma^2+o_0)}. \quad (29)$$

Case–6: $h_1 = h_3 = 0, h_0 = \frac{h_2^2g^2}{h_4(g^2+1)^2}$,

$h_2 < 0, h_4 > 0$

$a_0 = b_1 = 0$,

$$a_1 = \pm\sqrt{\frac{-h_4(6g_{11}h_2H_2+20g_{10}^2H_4)}{-h_2(h_2(g_{12}H_1+g_{13}H_3)+g_{12}H_6)}},$$

$$r = \pm\sqrt{\frac{-g_{10}^2H_4(h_2(H_1+H_3)+H_6) - h_2H_2(h_2(g_{10}^2H_1+g_{10}H_3)+g_{10}^2H_6)}{h_2^2(h_2(g_{12}H_1+g_{13}H_3)+g_{12}H_6)}},$$

$$H_5 = \frac{(h_2(g_{12}H_1+g_{13}H_3)+g_{12}H_6) \begin{pmatrix} 2g_{10}^2H_4(h_2(H_3-4H_1)+6H_6) + \\ 3h_2H_2\left(4g_{10}^2H_6-g_{10}^2h_2(2H_1-H_3)\right)\end{pmatrix}}{2(3g_{11}h_2H_2+10g_{10}^2H_4)^2} \quad (30)$$

with

$$\begin{cases} 103g_{10} = g^2 + 1, & g_{11} = 3g^4 + 2g^2 + 3, \\ g_{12} = g^4 + 14g^2 + 1, & g_{13} = g^4 - 6g^2 + 1. \end{cases}$$

We get a Jacobi elliptic solution as follows:

$$\psi(x,t) = \sqrt{\frac{(1-g_{10})(6g_{11}h_2H_2 + 20g_{10}^2H_4)}{g_{10}(h_2(g_{12}H_1 + g_{13}H_3) + g_{12}H_6)}} \times sn\left(\sqrt{\frac{-h_2}{g^2+1}}r(x-ct)\right)e^{i(-\tau x + \varsigma t + \gamma M(t) - t\gamma^2 + o_0)}, \quad (31)$$

$$\psi(x,t) = \sqrt{\frac{48h_2H_2 + 80H_4}{-2(h_2(16H_1 - 4H_3) + 16H_6)}} \times \tanh\left(\sqrt{\frac{-h_2}{g^2+1}}r(x-ct)\right)e^{i(-\tau x + \varsigma t + \gamma M(t) - t\gamma^2 + o_0)}. \quad (32)$$

Case–7: $h_1 = h_3 = 0, h_4 > 0$

$$a_0 = b_1 = 0,$$

$$a_1 = \pm\sqrt{\frac{2h_4(10h_2H_4 + 3(3h_2^2 - 4h_0h_4)H_2)}{(h_2^2 + 12h_0h_4)H_6 + h_2\left((h_2^2 + 12h_0h_4)H_1 + (h_2^2 - 8h_0h_4)H_3\right)}},$$

$$r = \pm\sqrt{\frac{-H_4(h_2(H_1 + H_3) + H_6) - H_2(h_2^2H_1 + h_2H_6 + (h_2^2 - 2h_0h_4)H_3)}{(h_2^2 + 12h_0h_4)H_6 + h_2\left((h_2^2 + 12h_0h_4)H_1 + (h_2^2 - 8h_0h_4)H_3\right)}},$$

$$H_5 = \frac{\begin{pmatrix} 2H_4(h_2(H_3 - 4H_1) + 6H_6) + \\ 3H_2\left(4h_2H_6 - (h_2^2 - 4h_0h_4)(2H_1 - H_3)\right)\end{pmatrix} \times \begin{pmatrix} (h_2^2 + 12h_0h_4)H_6 + \\ h_2\left((h_2^2 + 12h_0h_4)H_1 + (h_2^2 - 8h_0h_4)H_3\right)\end{pmatrix}}{2(10h_2H_4 + 3(3h_2^2 - 4h_0h_4)H_2)^2}. \quad (33)$$

We get a Weierstrass elliptic doubly periodic solution as follows:

$$\psi(x,t) = \pm\frac{\sqrt{18(10h_2H_4 + 3(3h_2^2 - 4h_0h_4)H_2)}}{\sqrt{(h_2^2 + 12h_0h_4)H_6 + h_2\left((h_2^2 + 12h_0h_4)H_1 + (h_2^2 - 8h_0h_4)H_3\right)}} \times \left[\frac{\wp(\zeta; f_2, f_3)}{6\wp(\zeta; f_2, f_3) + h_2}\right]e^{i(-\tau x + \varsigma t + \gamma M(t) - t\gamma^2 + o_0)}, \quad (34)$$

Where $f_2 = \frac{h_2^2}{12} + h_0h_4$ and $f_3 = \frac{h_2}{216}(36h_0h_4 - h_2^2)$ are called the Weierstrass elliptic function. Incidentally, this function also leads to bright soliton solutions with a special choice of the parameters.

4. Conclusions

This paper recovered optical soliton solutions to the concatenation model with Kerr law of SPM in the presence of white noise using an improved modified extended tanh-function approach. This unique approach gave way to a wide variety of soliton solutions to the model. The existence of such solitons is guaranteed with parameter constraints that are listed in the paper. A full spectrum of optical solitons has emerged, including the straddled solitons. The 1-soliton solutions have emerged through the intermediary Jacobi's elliptic functions, when the modulus of ellipticity approached the corresponding limit. A solution in terms of Weierstrass's elliptic function has also emerged that yields bright solitons under the special choice of the parameter combinations. One of the important observations made with the structure of these soliton solutions is that the white noise effect is confined to the phase component of the solitons. Thus, the amplitude component stays undeterred. This is a very important observation that is being made in the paper.

The results of the paper are, therefore, indeed very promising in the sense that the future of this project is very strong. The results are extendible to additional forms of SPM structure, such as the power-law. Subsequently, the dispersive concatenation model can be taken up in the future, both with Kerr law and the power law of SPM. These would lead to additional interesting results that are yet to be seen. Additionally, the two models can also be addressed in the presence of white noise but in the absence of SPM. The results of such studies are being conducted and are to be reported over time. Once these upcoming results are aligned with the various pre-existing results, they will be disseminated all across the board [17–20].

Disclosure

The authors claim there is no conflict of interest.

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Оптичне збурення солітонів для моделі конкатенації з мультиплікативним білим шумом за допомогою вдосконаленого модифікованого розширеного підходу на основі функції tanh

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Анотація. У цій статті відновлено одиничні та розділені оптичні солітони для моделі конкатенації за наявності мультиплікативного білого шуму за допомогою вдосконаленого модифікованого підходу на основі функції tanh. Також представлено параметричні обмеження для існування та підтримки таких солітонів. Спостерігається, що вплив білого шуму обмежується фазовою складовою відновлених солітонів.

Ключові слова: розсіяні оптичні солітони, інтегровність, модель конкатенації.